

# Elastic Stars in General Relativity I Foundations and Equilibrium Models

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## Abstract

The aim of this paper is twofold. First, we set up the theory of elastic matter sources within the framework of general relativity in a self-contained manner. The discussion is primarily based on the presentation of Carter and Quintana but also includes new methods and results as well as some modifications that in our opinion make the theory more modern and transparent. For instance, the equations of motion for the matter are shown to take a neat form when expressed in terms of the relativistic Hadamard elasticity tensor. Using this formulation we obtain simple formulae for the speeds of elastic wave propagation along eigendirections of the pressure tensor. Secondly, we apply the theory to static spherically symmetric configurations using a specific equation of state and consider models either having an elastic crust or core.

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## 1 Introduction

The presence of a solid crust in a neutron star is well established on theoretical grounds[1]. There are also speculations of solid cores due to the dominant strong interaction in the deep interiors of such stars. Although the presence of a solid crust is not thought to influence the mass or radius of a neutron star substantially there are still a number of other observational issues attributable to such a crust. The best observational data of what goes on inside a neutron star crust comes from pulsar timing measurements, occasionally showing sudden spin-ups in the otherwise very regular spin-down pattern. These events, called glitches, are thought to be due to interactions between the crust and superfluid neutrons penetrating the lattice see *e.g.* [2]. Another interesting feature observed in some pulsars is a smooth modulation of the timing measurements. This could be caused by free precession of a non-axisymmetric configuration of the crust with respect to the angular momentum axis. For discussions on this phenomenon see *e.g.* [3, 4]. Of course, a crust has many other interesting implications including for instance its effects on the (quasi) normal modes of the star. For a nonrotating newtonian perfect fluid star all axial modes are trivial in the sense that they

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are degenerate at zero frequency and hence correspond to non-oscillatory flows. The presence of the crust breaks the zero-frequency degeneracy of the axial modes as well as adding new modes due to the interface to the fluid core. For calculations and discussion within the Newtonian framework see [5, 6] and references therein. Axial modes of relativistic initially isotropic elastic stars have also been considered in some detail in [7]. For some very recent relativistic calculations within the Cowling approximation and using a Hookean equation of state see [8].

So far, all studies have been based on some kind of weak field approximation. Either it has been assumed that the gravitational field is weak or not coupled to matter oscillations, or that the deviation from an unsheared material state is small and can be treated linearly by a Hookean approximation. To be able to study nonlinear effects that fall outside such approximation schemes it is desirable to develop the general relativistic theory of elasticity to the point where it can be used in practical calculations. We intend for this work to be a step towards that goal.

Although classical newtonian elasticity theory has a long history, going back to the 17th century and Hooke's law, the development of a theory of elasticity adapted to general relativity took off rather late. The first demand for such a theory came in the late 50s with Weber's bar antenna for gravitational waves[9], necessitating an understanding of how these waves interact with elastic solids. For this application it is sufficient to use the linearised version of Einstein's equation with a background that can be treated newtonianly. Indeed, this weak-field approximation was used by Weber and was further developed in subsequent works by Dyson[10] and others. Following unsuccessful[11], partially successful[12] and successful yet rather coordinate-cluttered[13] attempts, a fully developed non-linear theory of elasticity adapted to general relativity was given in 1973 in a paper by Carter and Quintana[14] (hereinafter referred to as CQ), which still is a standard reference in the field. However, as later noted by Carter[15], the basic theoretical framework of their theory had already been given by Souriau[16], a work that largely had passed unnoticed. Also preceding CQ was Maugin who has made substantial contributions to the field, including the development of a theory of polarizable elastic media[17, 18], which should be important for realistic modeling of neutron stars since they are believed to have strong magnetic fields. Carter also gave a discussion of such matter in [15]. In more recent years, the general relativistic elasticity theory has been reconsidered by Magli and Kijowski[19, 20] who put emphasis on the gauge character of the theory, and by Christodoulou[21]. Very recently a number of existence and uniqueness theorems have been proved by Beig and Schmidt[22].

We would like to emphasise that the above bibliographical remarks by no means give a complete history of the subject of general relativistic elasticity theory. For further references see for instance [19, 23].

In this paper we set up the theory of general relativistic elasticity in a self-contained manner taking as our starting point the theory described in CQ. Although our treatment follows CQ rather closely and sometimes in detail, new methods and results are presented along the way. We do not want the reader to believe that we consider CQ to be superior to other sound presentations of the theory in any fundamental way. We simply choose it as a standard reference to avoid introducing yet another completely new set of notations and definitions into a field which already suffers from being a bit of a jungle when it comes to comparing results from different workers. Also, being relativists, we favor a formulation of the theory in terms of pure spacetime tensor equations, rather than formulations that keep material space indices and coordinates in the formulae.

The theoretical part of the paper is followed by an application to static spherically symmetric configurations. Having a thorough control and understanding of such models is essential as they provide the backgrounds for more elaborate applications, *e.g.* slowly rotating stars via the Hartle-Thorne formalism[24, 25] or stellar oscillations of various kinds. Spherically symmetric elastic matter models appears to have been first studied by Magli and Kijowski[26] and were later also investigated by Park[27]. However, in this paper our aim is to give a recipe for numerically producing physically relevant prestressed stellar models with speeds of wave propagation that are explicitly known. Such models will serve as backgrounds in subsequent investigations of stellar perturbations. The paper is organised as follows:

In section 2 we introduce and discuss the general concepts of the theory, based on the foundation that the matter field is a mapping from the four-dimensional spacetime manifold to an abstract three-dimensional manifold. The latter will be referred to as the material space, since its points may be thought of as the material particles. To insure that the theory is invariant under coordinate transformations on both manifolds,

the Lagrangian of the theory, which is directly given by the energy density, will be a function of scalars that can be formed by contractions of the inverse spacetime metric with covariant tensors that are pulled back from the material space. We feel that this is the most natural way to make the coordinate invariance inherent to the theory, as well as to have a clear distinction between the spacetime metric and the material space mapping as the gravitational and matter field variables of Einstein's equations with elastic source. This approach is slightly different from the one used in CQ, where the energy density is viewed as a function of the spatial part of the spacetime metric (which depends both on the spacetime metric itself as well as on the matter field), implicitly or explicitly contracted with contravariant tensors that can be identified as material space tensors. Using the spatial metric as a “variable” in this way would make it cumbersome to fix the spacetime metric and only consider variations of the matter field, which has been a source of critique from other workers (*cf.* the appendix of [28] and also [23]). However the critique is partially unjustified since it seems to be based on the assumption that the spatial metric is to be viewed as the elastic matter field.

Furthermore, the basic mathematical tools that we use are standard tools of differential geometry such as the pull-back and push-forward associated with the material space mapping. This means that we are using a somewhat lighter mathematical machinery than CQ who are explicitly making use of the isomorphism of the spacetime tangent subspaces orthogonal to the material flowlines and the tangent spaces of the material space manifold, as well as an extended version of the convected derivative first introduced by Oldroyd[13]. The reason that we are not discussing or using these concepts is not because we by any means feel that they are irrelevant, but because we, during the progress of this work, found that they were not needed for any of our purposes. For a thorough treatment of them, we instead refer the reader to CQ. The important results of this section is the general form of the stress-energy tensor and its specialization to the case when a fixed metric is the only material space tensor field which is used to set up the equation of state, an assumption which will be made throughout the paper unless explicitly stated otherwise.

In section 3 we discuss the matter equations of motion, *i.e.* the divergence-free condition for the stress-energy tensor. To this end we introduce a spatially projected connection on spacetime which can be viewed as the “pull-back” of the Levi-Cevita connection associated with the material space metric. This projected connection is given by a three index tensor field which we call the relativistic elasticity difference tensor. We then show that the Euler equations, to which the matter equations of motion reduces, can be put in an elegant form involving the four index relativistic Hadamard elasticity tensor and the relativistic elasticity difference tensor. To our knowledge, this form of the Euler equations is first given here. We also discuss how to generalise this method to more general equations of state involving material space tensor fields other than a metric.

In section 4 we derive various useful relations that involve the eigenvectors and eigenvalues of the pull-back of the material space metric. This leads us to introduce three scalars that we refer to as the linear particle densities which simply are defined as the square roots of the eigenvalues. The advantage of working with these linear particle densities is that we arrive at several formulae that closely resemble analogous formulae for perfect fluids when the latter are described in terms of the (volume) particle density.

In section 5 we derive the characteristic equation that determines the speed of wave propagation in relativistic elastic media. We do so by employing the standard Hadamard method of considering field discontinuities across a hypersurface, in a manner similar to Carter[29]. However, by starting out from our form of the nonlinear equations of motion we obtain a more direct link between the full nonlinear theory and the theory of wave propagation. The characteristic equation is explicitly solved in the case when the wave propagates in a principal (*i.e.* eigenvector) direction and we obtain a strikingly simple formula for the speed of the longitudinal waves, closely analogous to the formula for the speed of sound in perfect fluids. For the transversal waves, which have no analogue for perfect fluids, the propagation speed is found to be given by a simple algebraic formula involving the principal pressures and the linear particle densities. Although wave propagation in prestressed nonlinear elastic solids has been treated relativistically also by Maugin in a number of papers[28, 30, 31, 32], we feel that our approach and usage of the linear particle densities make the derivation as well as the resulting formulae much more transparent.

In section 6 we introduce a particular class of equations of state which much resemble the quasi-hookean type of equations of state described in CQ, the sole difference being that we use a different definition of the shear scalar which we feel is more convenient. Explicit formulae for the principal pressures and speeds of

sound propagation are given.

Section 7 is concerned with the degenerate case when two of the linear particle densities coincide. The reason why we pay particular attention to this nongeneric case is that any spherically symmetric configuration by necessity belongs to it.

In section 8 we set up Einstein's equations for static spherically symmetric configurations with an elastic matter source. The full set of equations of motion are found to form a (nonautonomous) system of three first order ordinary differential equations. This should be contrasted to the perfect fluid case for which the number of equations is two.

In section 9 we discuss the boundary conditions that static spherically symmetric configurations must satisfy. In particular we show that transitions between a material phase interior to an elastic solid form a one-parameter set of phase transitions parametrizable by the discontinuity in the tangential pressure. Physically speaking, the nature of such a phase transition is determined by the interaction between interior and exterior phases as well as the history of the configuration. However it is not directly given by the matching conditions of the Einstein equations. This is in contrast to the perfect fluid case where the phase transitions are given by the equation of state in a direct way.

We then proceed in section 10 to integrate some specific toy models of neutron stars using a relativistic polytropic equation of state as description of the unsheared behaviour of the matter. We let the shear modulus be proportional to the unsheared pressure and study the effects of on the one hand including a solid core, and on the other having a rigid crust. As expected the equilibrium models do not differ very much from the corresponding perfect fluid ones for realistic values of the shear modulus.

In the Appendix we discuss and motivate the class of equations of state introduced in section 6, including our unorthodox shear scalar definition. The discussion is based on a second order expansion of a general equation of state around an arbitrarily chosen unsheared state.

A few words on the notation and conventions used in this paper should be mentioned here. Tensors will be given in the abstract index notation in the manner described in Wald[33]. Abstract spacetime indices will be denoted by lowercase Latin letters ( $a, b, \dots$ ), whereas material space indices will be denoted by capital Latin letters ( $A, B, \dots$ ). Greek letters ( $\mu, \nu, \dots$ ) or ( $\Lambda, \Sigma, \dots$ ) will be used as indices referring to a given basis and for these the Einstein summation convention will not be used. In order to avoid cluttering formulae by explicitly writing out arguments we will use the convention that quantities depending only on the particle density will be denoted by a check accent. For example, the symbol  $\check{p}$  will denote unsheared pressure, to be explained subsequently. The metric is taken to have signature  $+2$  and the units are such that the speed of light is equal to unity and the Einstein equations take the form  $G_{ab} = \kappa T_{ab}$  with  $\kappa$  being given in arbitrary units.

## 2 Relativity

The theory of elastic matter sources in general relativity that we will use is based on the presentation of Carter and Quintana[14]. Since we expect that this theory is not generally known, we will start the discussion by outlining the basic ideas. To begin with, we introduce an abstract three-dimensional manifold, the *material space*,  $X$ , whose points are to be thought of as the material particles when going to the continuum limit. This space can in general be equipped with various types of tensor fields which give the structure of the material in a reference state. To begin with, the only tensor field that we introduce is, using capital Latin tensor indices on  $X$ , a *particle density form*  $n_{ABC} = n_{[ABC]}$ . This three-form is defined in such a way that when integrated over some volume in  $X$ , the result is simply the number of particles contained in that volume.

A *state of matter* in a material filled open four-dimensional submanifold  $M' \subseteq M$ , with  $M$  being the full spacetime manifold, is given by the spacetime metric  $g_{ab}$  restricted to  $M'$  and a  $C^1$  (at least) mapping

$$\psi : M' \rightarrow X. \tag{1}$$

This mapping  $\psi$  should be thought of as the matter field of the theory, analogous to a scalar field but with the difference that  $\psi$  maps to a three-dimensional space while the scalar field maps to a one-dimensional one. We shall always assume that the preimage  $\psi^{-1}(p)$  of all points  $p \in X' = \psi(M')$  is a single timelike

curve in  $M'$ , which is to be interpreted as the flowline of the particle represented by the point  $p$ . This way  $\psi$  gives rise to an identification of the three-dimensional manifold of particle flowlines with the material space submanifold  $X'$ . The push-forward  $\psi_*$  and pull-back  $\psi^*$ , taking contravariant tensors from  $M'$  to  $X'$  respectively covariant tensors from  $X'$  to  $M'$ , are important ingredients when setting up the theory. We shall use the convention that the push-forward of a contravariant spacetime tensor  $t^{a\cdots}$  will be denoted by  $t^{A\cdots}$  and similarly that the pull-back of a covariant material space tensor  $t_{A\cdots}$  will be denoted by  $t_{a\cdots}$ . In other words we simply replace the lower case Latin spacetime indices with capital Latin material space indices and vice versa. From the made assumptions about  $\psi$  it follows that the kernel of  $\psi_*$ , when viewed as a linear mapping of spacetime vectors  $v^a$  to material space vectors  $v^A$ , is comprised of the vectors that are tangent to the particle flowlines. It can be proved, using the commutation relations involving pull-back, push-forward, tensor contraction and Lie differentiation, that given a covariant spacetime tensor field  $t_{a\cdots}$ , there exists a fixed material space tensor field  $t_{A\cdots}$  whose pullback is  $t_{a\cdots}$  if and only if the following conditions hold for all vector fields  $v^a$  that are everywhere flowline tangential;

- All contractions of  $v^a$  with any index of  $t_{a\cdots}$  vanishes, *i.e.*  $t_{a\cdots}$  is completely flowline orthogonal.
- $t_{a\cdots}$  is Lie dragged by  $v^a$ ;  $\mathcal{L}_v t_{a\cdots} = 0$ .

The pulled back particle density form  $n_{abc} = \psi^* n_{ABC}$  is of fundamental importance for the theory and is used to define the flowline tangential *particle current*

$$n^a = \frac{1}{3!} \epsilon^{abcd} n_{bcd}, \quad (2)$$

where  $\epsilon_{abcd}$  is the spacetime volume form associated with  $g_{ab}$ . By construction,  $n^a$  is conserved,

$$\nabla_a n^a = 0, \quad (3)$$

which follows from the fact that exterior differentiation commutes with the pull-back operation, implying that  $n_{abc}$  is a closed spacetime three-form due to  $n_{ABC}$  trivially being a closed material space three-form. Noting that  $n^a$  by assumption is timelike, we set

$$n^a = n u^a, \quad n = \sqrt{-n^a n_a}, \quad u^a u_a = -1, \quad (4)$$

where  $n$  is the *particle density* and  $u^a$  the matter four-velocity. To proceed we introduce the spatial volume form

$$\epsilon_{abc} = \epsilon_{abcd} u^d. \quad (5)$$

From eq. (2) it follows that

$$n_{abc} = n \epsilon_{abc}, \quad (6)$$

which justifies the interpretation of  $n$  as the particle density.

An *equation of state* is supplied by giving the rest frame energy density  $\rho$  as a function of scalars that are formed by contracting the inverse metric  $g^{ab}$  with pulled back covariant material space tensor fields. The particle density may be used as one of the scalars, as indeed, according to eq. (6), it is given by the contraction

$$n^2 = \frac{1}{3!} n^{abc} n_{abc} = \frac{1}{3!} g^{ad} g^{be} g^{cf} n_{abc} n_{def}. \quad (7)$$

Once an equation of state is given, the stress-energy tensor takes the form

$$T_{ab} = -\rho g_{ab} + 2 \frac{\partial \rho}{\partial g^{ab}} = \rho u_a u_b + p_{ab}, \quad (8)$$

where

$$p_{ab} = 2 \frac{\partial \rho}{\partial g^{ab}} - \rho h_{ab}, \quad u^a p_{ab} = 0, \quad (9)$$

and

$$h_{ab} = u_a u_b + g_{ab}. \quad (10)$$

The above equations can be taken as the definition of the stress-energy tensor for elastic matter, but it is worth noting that  $T_{ab}$  can also be straightforwardly derived according to

$$T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}} \quad (11)$$

from the matter action

$$S_M = - \int dx^4 \sqrt{-g} \rho. \quad (12)$$

It will prove convenient to rewrite the energy density as  $\rho = n\epsilon$  where  $\epsilon$  is the energy per particle. For instance the pressure tensor  $p_{ab}$  is then given by the simple formula

$$p_{ab} = 2n \frac{\partial \epsilon}{\partial g^{ab}}, \quad (13)$$

where we have used that

$$\frac{\partial n}{\partial g^{ab}} = \frac{1}{2} n h_{ab}, \quad (14)$$

which follows, for instance, from eq. (7). Now, scalars that are formed on spacetime by contracting  $g^{ab}$  with pulled back tensors  $t_{a\dots} = \psi^* t_{A\dots}$  can alternatively be formed on the material space by making the analogous contractions of the pushed forward tensor  $g^{AB} = \psi_* g^{ab} = \psi_* h^{ab}$  with the tensors  $t_{A\dots}$ . Hence, at all points in  $X'$ , we may view the energy density  $\rho$  as a function of  $g^{AB}$  and fixed material space tensors  $t_{A\dots}$ . Taking this view we shall make the assumption that  $\epsilon$  has a minimum value  $\check{\epsilon}$  under variations of  $g^{AB}$  such that the particle density  $n$  is held fixed. Such a state will be referred to as an *unsheared* state. This allows us to introduce an  $n$ -dependent tensor  $\eta_{AB}$  on the material space such that  $g^{AC} \eta_{CB} = \delta^A_B$  when  $\epsilon = \check{\epsilon}$ . In other words, we define  $\eta_{AB}$  such that the minimum of  $\epsilon(g^{AB})$  occurs at  $g^{AB} = \eta^{-1AB}$ . If  $\check{\epsilon}$ , which is a function of  $n$  only, has a minimum at a certain particle density  $n = n_0$ , it follows that  $\epsilon$  has an absolute minimum under *all* variations of  $g^{AB}$ . This minimizing state is referred to as a *completely unstrained* or *completely relaxed* state. We shall not assume the existence of such a state simply because the material structure in the interiors of a neutron star may owe its existence to the high pressures there.

It is fairly straightforward to show that the pull-back of the Levi-Cevita volume form of  $\eta_{AB}$  coincides with the spatial volume form  $\epsilon_{abc}$ . Hence we let the volume form of  $\eta_{AB}$  be denoted  $\epsilon_{ABC}$  as we have  $\psi^* \epsilon_{ABC} = \epsilon_{abc}$ . Clearly, the relation between  $\epsilon_{ABC}$  and the particle density form  $n_{ABC}$  is analogous to eq. (6) for their pull-backs  $\epsilon_{abc}$  and  $n_{abc}$ , *i.e.*

$$n_{ABC} = n \epsilon_{ABC}. \quad (15)$$

It is worth keeping in mind that, despite the notation, the particle density form  $n_{ABC}$  is in fact a fixed material space tensor, independent of  $n$ . It is convenient to define a new tensor  $k_{AB}$ , conformal to  $\eta_{AB}$  and having  $n_{ABC}$  as its volume form. It follows from eq. (15) that this tensor is given by

$$k_{AB} = n^{2/3} \eta_{AB} \quad (16)$$

Denoting differentiation with respect to  $n$  by a prime we have

$$n \eta'_{AB} = -\frac{2}{3} \eta_{AB} + \tau_{AB}, \quad (17)$$

where

$$\tau_{AB} = n^{1/3} k'_{AB} \quad (18)$$

Now, since  $k^{-1AB} k'_{AB} = 0$  due to the fact that the determinant of  $k_{AB}$  is independent of  $n$  (since  $n_{ABC}$  is), we find that the trace of eq. (17) with respect to  $\eta^{-1AB} = n^{2/3} k^{-1AB}$  is

$$n \eta^{-1AB} \eta'_{AB} = -2 \quad (19)$$

Thus  $\tau_{AB}$ , called the *compressional distortion tensor*, is clearly the trace-free part of  $n\eta'_{AB}$  and therefore measures how the deformation of the unsheared crystal structure deviates from pure conformal rescaling when varying the particle density. It now follows from the definition (18) that  $k_{AB}$  is independent of  $n$  precisely when the material deforms conformally. In this paper we will mainly consider such materials, which include materials with isotropic but also, for instance, cubic unsheared structures. Consequently, we will have a fixed  $n$ -independent metric tensor field  $k_{AB}$  on  $X$ . We shall also make a further restricting assumption, namely that the energy per particle  $\epsilon$  is a function only of the independent invariants that can be formed from the flowline orthogonal mixed tensor

$$k^a_b = g^{ac}k_{cb}, \quad k_{ab} = \psi^*k_{AB}. \quad (20)$$

When acting with  $\partial/\partial g^{ab}$  on quantities that depend only on  $k^a_b$  we may reexpress this operator as

$$\frac{\partial}{\partial g^{ab}} = k_{c(a} \frac{\partial}{\partial k^b)_{c}}, \quad (21)$$

which follows from

$$\frac{\partial k^c_d}{\partial g^{ab}} = \delta^c_{(a} k_{b)d}. \quad (22)$$

Since  $k^a_b$  is restricted to be orthogonal to  $u^a$ , it has three independent scalar invariants, which for instance could be taken to be

$$I_1 = k^a_a, \quad I_2 = k^a_b k^b_a, \quad I_3 = k^a_b k^b_c k^c_a. \quad (23)$$

With  $n_{ABC}$  being the volume form of  $k_{AB}$ , it follows that the particle density  $n$  is in fact also a scalar invariant of  $k^a_b$ , given by

$$n^2 = \det(k^a_b) = \frac{1}{3!} (I_1^3 - 3I_1 I_2 + 2I_3). \quad (24)$$

Rather than using  $\{I_1, I_2, I_3\}$  as the set of three independent scalar invariants of  $k^a_b$ , it is both convenient and instructive to instead take the set to be comprised of  $n$  and two independent scalar invariants of the mixed tensor

$$\eta^a_b = h^{ac}\eta_{cb}, \quad \eta_{ab} = \psi^*\eta_{AB}, \quad (25)$$

whose deviation from  $h^a_b$  gives all information about deformations other than conformal compressions. From the relation  $k^a_b = n^{2/3}\eta^a_b$  it is clear that  $\det(\eta^a_b) = 1$ , whence  $\eta^a_b$  has two independent invariants rather than three, due to its invariance under conformal compressions  $h_{ab} \rightarrow C^{-2}h_{ab}$  under which  $k^a_b \rightarrow C^2k^a_b$  and  $n \rightarrow C^3n$ . We shall now obtain an alternative expression for the pressure tensor by using that the decomposition  $k^a_b = n^{2/3}\eta^a_b$  of  $k^a_b$  into a conformal factor  $n^{2/3}$  times a unit determinant mixed tensor  $\eta^a_b$  gives rise to the following splitting of  $\partial/\partial g^{ab}$  valid when acting on invariants (scalar or tensorial) of  $k^a_b$ :

$$\frac{\partial}{\partial g^{ab}} = \frac{1}{2}n h_{ab} \frac{\partial}{\partial n} + \eta_{c< a} \frac{\partial}{\partial \eta^{b> c}}, \quad (26)$$

where the angle brackets  $<, >$  around an index pair here denotes the symmetric and traceless part of that pair, where “traceless part” refers to  $h_{ab}$  and not the full spacetime metric  $g_{ab}$ . Explicitly, for a tensor  $t_{ab}$ , we have

$$t_{< ab >} = t_{(ab)} - \frac{1}{3}t^c_c h_{ab}. \quad (27)$$

In order to obtain eq. (26) we have used eq. (14) and

$$\frac{\partial \eta^c_d}{\partial g^{ab}} = h^c_{< a} \eta_{b> d} \quad (28)$$

which follows from  $\eta^c_d = n^{-2/3}k^c_d$  and eqs. (14) and (22). So, by specifying  $\epsilon$  as a function of  $n$  and  $\eta^a_b$ , the pressure tensor can be written as

$$p_{ab} = p h_{ab} + \pi_{ab}, \quad \pi^a_a = 0, \quad (29)$$

where

$$p = n^2 \frac{\partial \epsilon}{\partial n}, \quad (30)$$

$$\pi_{ab} = 2n \eta_{c < a} \frac{\partial \epsilon}{\partial \eta^{b > c}}. \quad (31)$$

This clearly displays that the dependence of  $\epsilon$  on the particle density  $n$  is directly related to the *isotropic pressure*  $p$ , while the dependence on the unimodular tensor  $\eta^a_b$  is directly related to the trace-free *anisotropy pressure tensor*  $\pi_{ab}$ . In particular, we see that the matter source can be interpreted as a perfect fluid iff the energy density is a function of the number density alone, the equation of state then being  $\rho = n\epsilon$ ,  $p = n^2 d\epsilon/dn$  where  $\epsilon$  coincides with  $\tilde{\epsilon}$ , the minimal energy per particle at fixed particle density.

### 3 Equations of motion for elastic matter

Let us now turn our attention to the equations of motion  $\nabla_b T^{ab} = 0$ , which can also be obtained by varying the matter action (12) with respect to the material space mapping  $\psi$ . For any stress-energy tensor of the form

$$T_{ab} = \rho u_a u_b + p_{ab}, \quad u^a u_a = -1, \quad u^a p_{ab} = 0, \quad (32)$$

the divergence free condition splits into its projection onto and orthogonal to  $u^a$ . In terms of the spatially projected connection  $D_a$  defined by its action on an arbitrary tensor  $t^{b \dots}_{c \dots}$  according to

$$D_a t^{b \dots}_{c \dots} = h^d_a h^b_e \dots h^f_c \dots \nabla_d t^{e \dots}_{f \dots}, \quad (33)$$

the equations of motion takes the form

$$\dot{\rho} + (\rho h^{ab} + p^{ab}) \Theta_{ab} = 0, \quad (34)$$

$$(\rho h^{ab} + p^{ab}) \dot{u}_b + D_b p^{ab} = 0, \quad (35)$$

where, as usual, a dot denotes covariant derivative along  $u^a$  and

$$\Theta_{ab} = D_{(a} u_{b)}. \quad (36)$$

For elastic matter, when the stress-energy tensor is given by an equation of state according to eqs. (8) and (9), it follows that eq. (34) is in fact automatically satisfied, leaving the  $u^a$  orthogonal *Euler equations* as the only nontrivial matter equations of motions.

When  $u^a$  is hypersurface orthogonal the projection tensor  $h_{ab}$  is the metric induced on these hypersurfaces. In this case the operator  $D_a$  is the Levi-Cevita connection associated with  $h_{ab}$ . In general, however,  $D_a$  cannot be viewed as a connection defined on a family of spacetime submanifolds, but it can nevertheless be viewed as the Levi-Cevita “pseudo-connection” of  $h_{ab}$  in the sense that the property

$$D_a h_{bc} = 0 \quad (37)$$

always holds. Since we are assuming that the material space is equipped with a fixed metric  $k_{AB}$ , its pullback  $k_{ab}$  provides an alternative metric on the tangent subspaces orthogonal to  $u^a$ . Furthermore, we may also carry over the Levi-Cevita connection  $\tilde{D}_A$  of  $k_{AB}$  into a differential operator  $\tilde{D}_a$  acting on spacetime tensors. To do so we declare that  $\tilde{D}_a$ , like  $D_a$ , should be a projected torsion-free connection with respect to the projection tensor  $h^a_b$ , *i.e.* there should exist a torsion-free connection  $\tilde{\nabla}_a$  on  $M$  such that the action of  $\tilde{D}_a$  on an arbitrary tensor field  $t^{b \dots}_{c \dots}$  is given by

$$\tilde{D}_a t^{b \dots}_{c \dots} = h^d_a h^b_e \dots h^f_c \dots \tilde{\nabla}_d t^{e \dots}_{f \dots}. \quad (38)$$

It follows that  $\tilde{D}_a$  is uniquely determined by  $D_a$  and a completely flowline orthogonal tensor field  $C^c_{ab}$  such that for all spacetime vector fields  $X^c$ , we have

$$\tilde{D}_a X^c = D_a X^c + C^c_{ab} X^b. \quad (39)$$



We will refer to the tensor field  $C^c_{ab}$  as the relativistic *elasticity difference tensor*, since the tensor field that gives the difference between two connections is sometimes called the difference tensor in the mathematical literature[34]. To relate  $\tilde{D}_a$  to  $\tilde{D}_A$  we impose that for any spacetime vector  $X^a$  and vector field  $Y^a$ , it should hold true that

$$\psi_*(X^b \tilde{D}_b Y^a) = X^B \tilde{D}_B Y^A, \quad X^B = \psi_* X^b, \quad Y^A = \psi_* Y^a. \quad (40)$$

This fixes  $C^c_{ab}$  to be the tensor field

$$C^c_{ab} = \frac{1}{2} k^{-1cd} (D_a k_{bd} + D_b k_{ad} - D_d k_{ab}), \quad (41)$$

with  $k^{-1cd}$  being the flowline orthogonal inverse of  $k_{ab}$ ,

$$k^{-1ac} k_{cb} = h^a_b, \quad k^{-1ab} u_b = k^{-1ba} u_b = 0. \quad (42)$$

Clearly, the tensor  $C^c_{ab}$  has the symmetry

$$C^c_{ab} = C^c_{(ab)}. \quad (43)$$

Moreover, as one might guess, the projected connection  $\tilde{D}_a$  satisfies

$$\psi_*(X^b \tilde{D}_b t^{a\dots}) = X^B \tilde{D}_B t^{A\dots}, \quad X^B = \psi_* X^b, \quad t^{A\dots} = \psi_* t^{a\dots} \quad (44)$$

$$\psi^*(\tilde{D}_B t_{A\dots}) = \tilde{D}_b t_{a\dots}, \quad t_{a\dots} = \psi^* t_{A\dots}. \quad (45)$$

In particular, eq. (45) implies

$$\tilde{D}_c k_{ab} = 0. \quad (46)$$

Letting  $t^{b\dots}_{c\dots}$  be a tensor field built up out of  $g^{ab}$  and  $k_{ab}$ , it follows from eq. (46) that

$$\tilde{D}_a t^{b\dots}_{c\dots} = \frac{\partial t^{b\dots}_{c\dots}}{\partial g^{de}} \tilde{D}_a g^{de} = \frac{\partial t^{b\dots}_{c\dots}}{\partial g^{de}} (\tilde{D}_a - D_a) g^{de} = 2 \frac{\partial t^{b\dots}_{c\dots}}{\partial g^{de}} C^{de}_a, \quad (47)$$

implying in turn that

$$D_a t^{b\dots}_{c\dots} = \tilde{D}_a t^{b\dots}_{c\dots} - (\tilde{D}_a - D_a) t^{b\dots}_{c\dots} = 2 \frac{\partial t^{b\dots}_{c\dots}}{\partial g^{de}} C^{de}_a - t^{d\dots}_{c\dots} C^b_{ad} - \dots + t^{b\dots}_{d\dots} C^d_{ac} + \dots \quad (48)$$

In particular, it follows that if the energy per particle  $\epsilon$  is a function only of scalar invariants of  $k^a_b$ , the spatial divergence of the pressure tensor that occurs in Euler's equations (35) can be reexpressed as

$$D_b p^{ab} = A^{ab}_{cd} C^{cd}_b, \quad (49)$$

where  $A^{ab}_{cd}$  is the relativistic *Hadamard elasticity tensor*

$$A^{ab}_{cd} = 2 \frac{\partial p^{ab}}{\partial g^{cd}} - p^{ab} h_{cd} - h^a_c p^b_d. \quad (50)$$

The first two terms on the right hand side of the above equation is the relativistic *elasticity tensor*,

$$E^{ab}_{cd} = 2 \frac{\partial p^{ab}}{\partial g^{cd}} - p^{ab} h_{cd}, \quad (51)$$

which, using eq. (14), can be written in the more compact form

$$E^{ab}_{cd} = 2n \frac{\partial}{\partial g^{cd}} \left( \frac{p^{ab}}{n} \right). \quad (52)$$

With all indices raised, it can also be expressed in the familiar form (*cf.* CQ)

$$E^{abcd} = 4n \frac{\partial^2 \epsilon}{\partial g_{ab} \partial g_{cd}}, \quad (53)$$

which follows from eq. (13) and

$$\frac{\partial}{\partial g_{ab}} = -g^{ac} g^{bd} \frac{\partial}{\partial g^{cd}}. \quad (54)$$

Both  $A^{abcd}$  and  $E^{abcd}$  are flowline orthogonal tensors symmetric under interchange of the index pairs  $ab$  and  $cd$ , *i.e.*

$$A^{abcd} = A^{cdab}, \quad E^{abcd} = E^{cdab}. \quad (55)$$

In addition, the elasticity tensor has the symmetry

$$E^{abcd} = E^{(ab)(cd)}. \quad (56)$$

The bottom line of this discussion is that the equations of motion are given by the Euler equations (35) alone, which, using eq. (49), can be expressed as

$$(\rho h^{ab} + p^{ab}) \dot{u}_b + A^{abcd} C_{cbd} = 0. \quad (57)$$

This fact will be used in section 5 when we examine wave propagation in elastic media.

Before closing this section we would like to make a digression and argue that the Euler equations in the form (57), which to our knowledge has not appeared before in the literature, is valid under more general circumstances than were assumed when deriving it above. The assumption more restrictive than necessary was that the energy per particle  $\epsilon$  is a function only of scalar invariants of  $k^a_b = g^{ac} k_{cb}$  with  $k_{cb}$  being the pull-back of a fixed ( $n$ -independent) material space metric  $k_{CB}$ . Indeed, the derivation of eq. (57) can be immediately generalised to the case when there exists a torsion-free connection  $\tilde{D}_A$  on  $X$  such that

$$\tilde{D}_A t_{B\dots} = 0, \quad (58)$$

for all fixed covariant tensors  $t_{A\dots}$  on  $X$  which contribute to the equation of state. This connection can be carried over into a projected connection  $\tilde{D}_a$  on  $M$  in exactly the same manner as before, *i.e.* through eqs. (38) - (40). Hence  $\tilde{D}_a$  will again be uniquely determined in terms of  $D_a$  and an elasticity difference tensor  $C^c_{ab}$  which, due to the requirement that  $\tilde{D}_A$  be torsion-free, has the index symmetry (43). Since eqs. (44) and (45) will hold also in this case, the pull-back of eq. (58) is of course

$$\tilde{D}_a t_{b\dots} = 0, \quad t_{b\dots} = \psi^* t_{B\dots}, \quad (59)$$

which implies that the remaining steps that lead up to the Euler equations in the form (57) are still valid.

The most physically relevant situation when this generalization applies is when the reference structure of the material space is completely given by three *crystal axis vectors*

$$E_\Lambda^A, \quad \Lambda = 1, 2, 3, \quad (60)$$

which, mathematically, are three linearly independent vector fields assumed to be commuting,

$$[E_\Lambda, E_\Sigma]^A = 0. \quad (61)$$

Physically, the crystal axis vectors are to be thought of as displacement vectors pointing from one lattice point to three of its nearest neighbouring lattice points, with the three neighbours chosen in an ordered way such that the crystal axis vectors become smooth vector fields when going to the continuum limit. The two-dimensional surfaces that, due to the commutation requirement (61), are guaranteed to be formed by any pair of crystal axis vectors ( $E_\Lambda^A, E_\Sigma^A$ ) with  $\Lambda \neq \Sigma$  are consequently to be thought of as crystal planes. The basis of one-form fields  $\omega^\Lambda_A$  that are dual to the crystal lattice vectors, *i.e.* satisfying

$$\omega^\Lambda_A E_\Sigma^A = \delta^\Lambda_\Sigma, \quad (62)$$

will here be called the *reciprocal lattice one-forms*, as they are in direct correspondence with the reciprocal lattice vectors of classical lattice/crystal theory (*cf.* [35]) where, of course, no distinction is made between vectors and one-forms. However, it should be stressed that a factor  $2\pi$  should be inserted on the right hand side of eq. (62) to conform to the standard definition of reciprocal lattice vectors. Now, if each lattice point is associated with a number  $N_f$  of particles belonging to the species that one wishes to consider as fundamental (*e.g.* baryons), the particle density form is simply

$$n_{ABC} = 3! N_f \omega^1_{[A} \omega^2_B \omega^3_{C]}. \quad (63)$$

Indeed, this three-form associates a volume  $N_f$  to a cell spanned by the crystal axis vectors,

$$E_1^A E_2^B E_3^C n_{ABC} = N_f, \quad (64)$$

and, clearly, such cells are in one-to-one correspondence with the lattice points.

If we assume that no further tensor fields are introduced on  $X$ , the energy per particle  $\epsilon$  can only depend on the scalars

$$I^{\Lambda\Sigma} = g^{ab} \omega^\Lambda_a \omega^\Sigma_b, \quad \omega^\Lambda_a = \psi^* \omega^\Lambda_A, \quad (65)$$

of which six are independent since  $I^{\Lambda\Sigma} = I^{\Sigma\Lambda}$ . We thus would like to introduce a torsion-free connection  $\tilde{D}_A$  for which all three reciprocal lattice one-forms are covariantly constant,

$$\tilde{D}_A \omega^\Lambda_B = 0. \quad (66)$$

This is possible because eqs. (61) and (62) imply that these one-forms are closed, thus making the anti-symmetric part of eq. (66) identically satisfied. Consequently, eq. (66) implies that  $\tilde{D}_A$  is the unique flat torsion-free connection for which the coordinate frame connection coefficients vanishes when using coordinates  $x^\Lambda$  such that  $\omega^\Lambda_A = \partial_A x^\Lambda$ . The elasticity difference tensor appearing in the Euler equations (57) will in this case take the very simple form

$$C^c_{ab} = \sum_{\Lambda=1}^3 E_\Lambda^c D_a \omega^\Lambda_b, \quad (67)$$

where the spacetime vector fields  $E_\Lambda^a$  can be identified with the material space vector fields  $E_\Lambda^A$  since they are defined by

$$\omega^\Lambda_a E_\Sigma^a = \delta^\Lambda_\Sigma, \quad u_a E_\Lambda^a = 0. \quad (68)$$

Indeed, they of course satisfy

$$\psi_* E_\Lambda^a = E_\Lambda^A. \quad (69)$$

To arrive at eq. (67) we have used that

$$D_{[a} \omega^\Lambda_{b]} = 0, \quad (70)$$

which follows from the antisymmetric part of eq. (66) and guarantees that the symmetry property (43) holds. The result that the matter equations of motion in this case reduces to eq. (57) with  $C^c_{ab}$  given by eq. (67) is independent of whether or not the unsheared crystal structure is conformally rescaled under variation of the particle density, *i.e.* whether or not the compressional distortion tensor  $\tau_{AB}$  is identically vanishing. However, if we do assume that  $\tau_{AB} = 0$ , we have a preferred fixed ( $n$ -independent) material space metric which by necessity is of the form

$$k_{AB} = \sum_{\Lambda, \Sigma=1}^3 k_{\Lambda\Sigma} \omega^\Lambda_A \omega^\Sigma_B, \quad (71)$$

where the metric components  $k_{\Lambda\Sigma}$  are constants subject to the condition

$$\det(k_{\Lambda\Sigma}) = N_f^2, \quad (72)$$

thus making  $n_{ABC}$  the associated volume form of  $k_{AB}$ . The reason why  $k_{AB}$  can only be of this form is simply because there is no other way to form a fixed metric on  $X$  without introducing further tensor fields contributing to the equation of state. The constants  $k_{\Lambda\Sigma}$  give the lattice structure of the material in its unsheared state. For instance, the special case

$$k_{\Lambda\Sigma} = N_f^{2/3} \delta_{\Lambda\Sigma} \quad (73)$$

corresponds to cubic lattices. The three independent scalar invariants that can be formed from  $k^a_b$  will obviously be functions of the six independent scalar invariants  $I^{\Lambda\Sigma}$ . If  $\epsilon$  does not depend on any combinations of the latter that cannot be formed out of  $k^a_b$ , the equation of state belongs to the class that this paper is mainly concerned with. Incidentally, this serves as a good motivation for taking the material space metric  $k_{AB}$  to be flat.

## 4 Eigenvalue and eigenvector formulation

In this section we shall explore the fact that whichever scalar invariants of  $k^a_b$  we are using to parametrise the equation of state by, they can always be expressed in terms of the eigenvalues of  $k^a_b$  and, consequently, so can the energy per particle  $\epsilon$ . Since  $k^a_b$  is flowline orthogonal and since  $h_{ab}$  (viewed as a three-dimensional metric) is positive definite, the eigenvectors of  $k^a_b$  corresponding to distinct eigenvalues are automatically orthogonal, while eigenvectors with the same eigenvalue can be chosen orthogonal. Moreover, since  $k_{ab}$  is also positive definite, all eigenvalues of  $k^a_b$  will be positive and we shall write them as  $n_\mu^2$ ,  $\mu = 1, 2, 3$ , in terms of their positive square roots  $n_\mu$  which, in a loose sense, represent one-dimensional particle densities in the principal directions, *i.e.* along the eigenvectors. We hence refer to these quantities as *principal linear particle densities*, or *linear particle densities* for short. According to eq. (24), the particle density  $n$  is simply the product of the linear particle densities, *i.e.*

$$n = n_1 n_2 n_3. \quad (74)$$

Using an orthonormal basis  $\{e_\mu^a\}$  of eigenvectors of  $k^a_b$ , implying that

$$g_{ab} = -u_a u_b + \sum_{\mu=1}^3 e_{\mu a} e_{\mu b}, \quad (75)$$

$$k_{ab} = \sum_{\mu=1}^3 n_\mu^2 e_{\mu a} e_{\mu b}, \quad (76)$$

it follows from eq. (21) that the operator  $\partial/\partial g^{ab}$ , when acting on scalar invariants of  $k^a_b$ , takes the particularly simple form

$$\frac{\partial}{\partial g^{ab}} = \frac{1}{2} \sum_{\mu=1}^3 e_{\mu a} e_{\mu b} n_\mu \frac{\partial}{\partial n_\mu}. \quad (77)$$

The pressure tensor then becomes

$$p_{ab} = \sum_{\mu=1}^3 p_\mu e_{\mu a} e_{\mu b}, \quad (78)$$

where

$$p_\mu = n n_\mu \frac{\partial \epsilon}{\partial n_\mu}. \quad (79)$$

Hence  $p^a_b$  has the same eigenvectors as  $k^a_b$ , while the eigenvalues  $p_\mu$ , which we call the *principal pressures*, are given by eq. (79).

We now proceed to calculate the elasticity tensor according to eq. (51). To do so we need to know how to differentiate the eigenvectors with respect to  $g^{ab}$ . Therefore we let  $\partial/\partial g^{ab}$  act on the relations

$$g_{cd} e_\rho^c e_\sigma^d = \delta_{\rho\sigma}, \quad (80)$$

$$k_{cd} e_\rho^c e_\sigma^d = n_\rho^2 \delta_{\rho\sigma}. \quad (81)$$

This results in, respectively,

$$e_{\rho c} \frac{\partial e_\sigma^c}{\partial g^{ab}} + e_{\sigma c} \frac{\partial e_\rho^c}{\partial g^{ab}} = e_{\rho(a} e_{\sigma b)} \quad (82)$$

$$n_\rho^2 e_{\rho c} \frac{\partial e_\sigma^c}{\partial g^{ab}} + n_\sigma^2 e_{\sigma c} \frac{\partial e_\rho^c}{\partial g^{ab}} = n_\rho^2 e_{\rho a} e_{\rho b} \delta_{\rho\sigma} \quad (83)$$

Assuming that  $n_\rho \neq n_\sigma$  when  $\rho \neq \sigma$ , we can solve for all projections of the differentiated basis vectors. The result reads

$$e_{\rho c} \frac{\partial e_\rho^c}{\partial g^{ab}} = \frac{1}{2} e_{\rho(a} e_{\rho b)} \quad (84)$$

$$e_{\sigma c} \frac{\partial e_\rho^c}{\partial g^{ab}} = \frac{n_\rho^2}{n_\rho^2 - n_\sigma^2} e_{\rho(a} e_{\sigma b)} \quad \rho \neq \sigma, \quad (85)$$

which implies

$$\frac{\partial e_\rho^c}{\partial g^{ab}} = \frac{1}{2} e_{\rho c} e_{\rho a} e_{\rho b} + \sum_{\sigma \neq \rho} \frac{n_\rho^2}{n_\rho^2 - n_\sigma^2} e_{\sigma c} e_{\rho(a} e_{\sigma b)}. \quad (86)$$

Making use of eq. (77) when  $\partial/\partial g^{ab}$  acts on the scalars  $p_\rho$ , we find

$$E_{ab}{}^{cd} = \sum_{\rho=1}^3 \left[ 2p_\rho \left( e_{\rho c} \frac{\partial e_\rho^d}{\partial g^{ab}} + e_{\rho d} \frac{\partial e_\rho^c}{\partial g^{ab}} \right) + \sum_{\sigma=1}^3 \left( n_\sigma \frac{\partial p_\rho}{\partial n_\sigma} - p_\rho \right) e_{\sigma a} e_{\sigma b} e_{\rho c} e_{\rho d} \right]. \quad (87)$$

Using eq. (86) for the derivatives of the basis vectors and raising the indices  $a$  and  $b$ , we obtain our final form for the elasticity tensor:

$$E^{abcd} = \sum_{\rho=1}^3 \left[ 2p_\rho e_\rho^a e_\rho^b e_\rho^c e_\rho^d + 2 \sum_{\sigma \neq \rho} \frac{n_\rho^2 p_\rho - n_\sigma^2 p_\sigma}{n_\rho^2 - n_\sigma^2} e_\rho^{(a} e_\sigma^{b)} e_\rho^{(c} e_\sigma^{d)} + \sum_{\sigma=1}^3 \left( n_\sigma \frac{\partial p_\rho}{\partial n_\sigma} - p_\rho \right) e_\sigma^a e_\sigma^b e_\rho^c e_\rho^d \right]. \quad (88)$$

Note that the symmetry property  $E^{abcd} = E^{cdab}$  requires that the quantities

$$n_\sigma \frac{\partial p_\rho}{\partial n_\sigma} - p_\rho \quad (89)$$

are symmetric under  $\rho \leftrightarrow \sigma$ . This is indeed the case, since, according to eq. (79), they can be expressed as

$$n_\sigma \frac{\partial p_\rho}{\partial n_\sigma} - p_\rho = n n_\rho n_\sigma \frac{\partial^2 \epsilon}{\partial n_\rho \partial n_\sigma} + n n_\rho \frac{\partial \epsilon}{\partial n_\rho} \delta_{\rho\sigma}. \quad (90)$$

The Hadamard elasticity tensor is now

$$A^{abcd} = \sum_{\rho=1}^3 \left\{ \sum_{\sigma=1}^3 n_\sigma \frac{\partial p_\rho}{\partial n_\sigma} e_\sigma^a e_\sigma^b e_\rho^c e_\rho^d + 2 \sum_{\sigma \neq \rho} \left[ \frac{n_\rho^2 p_\rho - n_\sigma^2 p_\sigma}{n_\rho^2 - n_\sigma^2} e_\rho^{(a} e_\sigma^{b)} e_\rho^{(c} e_\sigma^{d)} - p_\rho e_\sigma^a e_\sigma^b e_\rho^c e_\rho^d \right] \right\}. \quad (91)$$

Let us now return to the decomposition  $k^a{}_b = n^{2/3} \eta^a{}_b$ , which, as discussed in section 2, in a natural way splits  $p_{ab}$  into its trace part  $ph_{ab}$  and its trace-free part  $\pi_{ab}$ . Doing so, we do no longer wish to parametrise

the equation of state by the linear particle densities  $n_\mu$ , but rather by the particle density  $n$  and variables that are closely related to the eigenvalues of  $\eta^a_b$ . Clearly,  $\eta^a_b$  has the same orthonormal eigenvectors  $e_\mu^a$  as  $k^a_b$ , the eigenvalues being  $(\alpha_1^2, \alpha_2^2, \alpha_3^2)$ , say, with

$$\alpha_\mu = \frac{n_\mu}{n^{1/3}} = \left( \frac{n_\mu^2}{n_{\mu+1}n_{\mu+2}} \right)^{1/3} = \left( \frac{z_{\mu+2}}{z_{\mu+1}} \right)^{1/3}, \quad \alpha_1\alpha_2\alpha_3 = 1. \quad (92)$$

where

$$z_\mu = \frac{n_{\mu+1}}{n_{\mu+2}} = \frac{\alpha_{\mu+1}}{\alpha_{\mu+2}}, \quad z_1z_2z_3 = 1, \quad (93)$$

and the cyclic rule  $\mu + 3 = \mu$  is being used. We find it useful to parametrise scalar invariants of  $\eta^a_b$  by the  $z_\mu$ 's. From  $n = n_1n_2n_3$  and  $z_\mu = n_{\mu+1}/n_{\mu+2}$  it follows that

$$n_\mu \frac{\partial}{\partial n_\mu} = n \frac{\partial}{\partial n} + z_{\mu+2} \frac{\partial}{\partial z_{\mu+2}} - z_{\mu+1} \frac{\partial}{\partial z_{\mu+1}}, \quad (94)$$

which via eqs. (26) and (77) gives

$$\eta_{c<a} \frac{\partial}{\partial \eta^{b>c}} = \frac{1}{2} \sum_{\mu=1}^3 e_{\mu a} e_{\mu b} \left( z_{\mu+2} \frac{\partial}{\partial z_{\mu+2}} - z_{\mu+1} \frac{\partial}{\partial z_{\mu+1}} \right). \quad (95)$$

It hence follows from eq. (31) that the anisotropy pressure tensor is

$$\pi_{ab} = \sum_{\mu=1}^3 \pi_\mu e_{\mu a} e_{\mu b}, \quad (96)$$

$$\pi_\mu = n \left( z_{\mu+2} \frac{\partial \epsilon}{\partial z_{\mu+2}} - z_{\mu+1} \frac{\partial \epsilon}{\partial z_{\mu+1}} \right), \quad (97)$$

where the eigenvalues  $\pi_\mu$  obey  $\sum_{\mu=1}^3 \pi_\mu = 0$  and are related to the principal pressures  $p_\mu$  and the isotropic pressure  $p$  by

$$p_\mu = p + \pi_\mu. \quad (98)$$

Note that the combinations  $z_{\mu+2} \partial/\partial z_{\mu+2} - z_{\mu+1} \partial/\partial z_{\mu+1}$  all annihilate  $z_1z_2z_3$ , which means that the condition  $z_1z_2z_3 = 1$  may be used to simplify anything these operators act on.

## 5 Wave propagation in elastic materials

Our discussion of the speed of sound in elastic materials will be based on the discontinuity formalism introduced by Hadamard about a century ago. The notation will follow Carter's as presented in [29], but as our starting point we will take the Euler equations expressed in terms of the Hadamard tensor, see eq. (57).

We consider a hypersurface  $\Sigma$  representing a wave front across which second order derivatives of the material space mapping  $\psi$  are allowed to be discontinuous, whereas continuity is assumed for all first order derivatives of  $\psi$  as well as of the spacetime metric. In other words the assumption is that  $\psi$  and  $g_{ab}$  are both  $C^1$ . The metric may be assumed to have a higher degree of differentiability, but this is irrelevant to the discussion. The continuity of the first order derivatives of  $\psi$  implies that all tensors on  $M$  which are pulled back from  $X$  are continuous. Since it thus follows that the particle four-velocity  $u^a$  is continuous, *i.e.* well-defined on  $\Sigma$ , the wave front propagates at a well-defined speed  $v$  with respect to the particle flow, see fig. 1.

In order to derive an equation governing the propagation of the wave front we start by noting that the tensors  $\rho h^{ab} + p^{ab}$  and  $A^{abcd}$  occurring in Euler's equations are continuous, whereas the acceleration  $\dot{u}^a$  and

the elasticity difference tensor  $C^c_{ab}$  may be discontinuous across  $\Sigma$ . Denoting the discontinuity of a quantity by surrounding square brackets we therefore have

$$(\rho h^{ab} + p^{ab})[\dot{u}_b] + A^{abcd}[C_{cbd}] = 0. \quad (99)$$

Following Carter, we set

$$[\dot{u}_a] = \alpha \iota_a, \quad \iota^a \iota_a = 1, \quad (100)$$

where  $\alpha$  is the *amplitude* and  $\iota_a$  is the *polarization vector* of the wave front. Since the discontinuity of the gradient of a scalar is always normal to the wave front and the Christoffel symbols are continuous due to the continuity of the first order metric derivatives it follows that for any tensor  $t^{a\cdots b\cdots}$  defined across  $\Sigma$ , there exists a tensor  $\tau^{a\cdots b\cdots}$  defined on  $\Sigma$  such that

$$[\nabla_c t^{a\cdots b\cdots}] = \tau^{a\cdots b\cdots} \lambda_c, \quad \lambda_a = \nu_a - v u_a, \quad (101)$$

where  $\nu_a$  is the *propagation vector* of the wave front, assumed to be normalised,

$$\nu^a \nu_a = 1. \quad (102)$$

It follows that there are tensors  $\kappa^a$  and  $\zeta_{ab}$  such that

$$[\nabla_b u^a] = \kappa^a \lambda_b \quad (103)$$

$$[\nabla_c k_{ab}] = \zeta_{ab} \lambda_c. \quad (104)$$

By contracting (103) with  $u^b$  and using eq. (100) we find

$$\kappa^a = \frac{\alpha}{v} \iota^a \quad (105)$$

Moreover, the discontinuity of  $\mathcal{L}_u k_{ab} = u^c \nabla_c k_{ab} + 2k_{c(a} \nabla_{b)} u^c = 0$  gives

$$\zeta_{ab} = -2 \frac{\alpha}{v^2} \iota^c k_{c(a} \lambda_{b)}, \quad (106)$$

which in turn implies

$$[C^c_{ab}] = -\frac{\alpha}{v^2} \iota^c \nu_a \nu_b. \quad (107)$$

Finally, substituting eqs. (100) and (107) into eq. (99) we arrive at

$$[v^2(\rho h^{ac} + p^{ac}) - Q^{ac}] \iota_c = 0, \quad (108)$$

where  $Q^{ac} = Q^{(ac)}$  is the relativistic *Fresnel tensor*, given by

$$Q^{ac} = A^{abcd} \nu_b \nu_d. \quad (109)$$

We have now shown that the Euler equations constrain the quantities  $\nu_a$ ,  $\iota_a$  and  $v$  algebraically according to eq. (108). In realistic situations the combination  $\rho h^{ab} + p^{ab}$ , viewed as a three-dimensional tensor, is expected to be non-degenerate. Clearly, the propagation vector  $\nu_a$  can then be specified freely since for each  $\nu_a$  eq. (108) turns into a three-dimensional eigenvalue problem, with  $v^2$  being the eigenvalue and  $\iota_a$  the eigenvector. Since eq. (108) is a three-dimensional equation, it has three independent solutions. Moreover it may be noted that as a consequence of the symmetry of  $Q^{ab}$ , it follows that any two polarization vectors  $^{(1)}\iota_a$  and  $^{(2)}\iota_a$  corresponding to distinct eigenvalues  $^{(1)}v^2$  and  $^{(2)}v^2$  are orthogonal with respect to  $\rho h^{ab} + p^{ab}$ . For a general propagation vector  $\nu^a$ , the Fresnel tensor can be written as

$$\begin{aligned} Q^{ac} &= \sum_{\rho=1}^3 \left\{ \sum_{\sigma=1}^3 \nu_\rho \nu_\sigma n_\sigma \frac{\partial p_\rho}{\partial n_\sigma} e_\rho^{(a} e_{\sigma}^{c)} + \sum_{\sigma \neq \rho} \frac{n_\sigma^2 (p_\sigma - p_\rho)}{n_\sigma^2 - n_\rho^2} \left[ \nu_\rho^2 e_{\sigma}^{(a} e_{\sigma}^{c)} + \nu_\rho \nu_\sigma e_\rho^{(a} e_{\sigma}^{c)} \right] \right\} \\ &= \sum_{\rho=1}^3 \left[ \nu_\rho^2 n_\rho \frac{\partial p_\rho}{\partial n_\rho} + \sum_{\sigma \neq \rho} \nu_\sigma^2 \frac{n_\rho^2 (p_\rho - p_\sigma)}{n_\rho^2 - n_\sigma^2} \right] e_\rho^{(a} e_{\rho}^{c)} \\ &\quad + \sum_{\rho=1}^3 \sum_{\sigma \neq \rho} \nu_\rho \nu_\sigma \left[ n_\sigma \frac{\partial p_\rho}{\partial n_\sigma} + \frac{n_\sigma^2 (p_\sigma - p_\rho)}{n_\sigma^2 - n_\rho^2} \right] e_\rho^{(a} e_{\sigma}^{c)}, \end{aligned} \quad (110)$$

where  $\nu_\rho = e_\rho^a \nu_a$ . We shall only solve the characteristic equation explicitly for the case when the propagation vector  $\nu_a$  coincides with one of the eigenvectors, say  $e_{\mu a}$ . The Fresnel tensor then reduces to

$$Q^{ac} = n_\mu \frac{\partial p_\mu}{\partial n_\mu} e_\mu^a e_\mu^c + \sum_{\sigma \neq \mu} \frac{n_\sigma^2 (p_\sigma - p_\mu)}{n_\sigma^2 - n_\mu^2} e_\sigma^a e_\sigma^c. \quad (111)$$

Since

$$\rho h^{ac} + p^{ac} = \sum_{\sigma=1}^3 (\rho + p_\sigma) e_\sigma^a e_\sigma^c, \quad (112)$$

it follows that the characteristic equation (108) has one solution corresponding to a purely longitudinally polarised mode, given by

$$v^2 = v_{\mu||}^2 = (\rho + p_\mu)^{-1} n_\mu \frac{\partial p_\mu}{\partial n_\mu}, \quad \iota^a = \nu^a = e_\mu^a. \quad (113)$$

The remaining two independent solutions are

$$v^2 = v_{\mu\perp\nu}^2 = (\rho + p_\nu)^{-1} \frac{n_\nu^2 (p_\nu - p_\mu)}{n_\nu^2 - n_\mu^2}, \quad \iota^a = e_\nu^a, \quad \iota^a \nu_a = 0. \quad (114)$$

These modes correspond to transversal waves whose polarization vector coincides with one of two eigenvectors  $e_\nu^a$  orthogonal to the propagation eigenvector  $e_\mu^a$ . Remarkably, the formula (113) for longitudinal waves in eigenvector directions exactly mimics the corresponding formula for the speed of sound  $c_s$  in a perfect fluid with equation of state  $\rho = \rho(n)$ ,  $p = p(n)$ , namely

$$c_s^2 = (\rho + p)^{-1} n \frac{dp}{dn}. \quad (115)$$

For perfect fluids the quantity  $\beta = n dp/dn$  can be identified with the compressibility modulus, or bulk modulus. Thus, since a perfect fluid can be viewed as a special case of an elastic material, we take the analogy further by introducing the *principal compressibility moduli*  $\beta_\mu$  according to

$$\beta_\mu = n_\mu \frac{\partial p_\mu}{\partial n_\mu}, \quad (116)$$

in terms of which the speeds of the eigenvector directed longitudinal waves (sound waves) are given by

$$v_{\mu||}^2 = \frac{\beta_\mu}{\rho + p_\mu}. \quad (117)$$

It is worth noting that  $v_{\mu||}^2$  can also be rewritten as

$$v_{\mu||}^2 = \frac{\partial p_\mu}{\partial n_\mu} / \frac{\partial \rho}{\partial n_\mu}, \quad (118)$$

analogous to the well-known perfect fluid formula  $c_s^2 = dp/d\rho$ .

Our resulting expressions for the nine principal speeds  $v_{\mu||}$  and  $v_{\mu\perp\nu}$  are relevant to compare with those obtained by Maugin in [28], since in both cases it is assumed that the equation of state is set up using a fixed metric as the only material space structure. We could go through some lengths to establish the precise connection between the two sets of formulae, but refrain from doing so since it would only lead to the introduction and explanation of new notation rather than to something illuminating for anyone but a very specialised reader. In [28] the propagation speeds are given in terms of quantities that explicitly refers to a particular way of expressing the pressure tensor  $p_{ab}$  by making use of the Cayley-Hamilton theorem. In contrast, our recipe for calculating  $v_{\mu||}$  and  $v_{\mu\perp\nu}$  from the equation of state  $\epsilon = \epsilon(n_1, n_2, n_3)$  is very simple and direct which should make it favourable for practical applications.



Although we do not treat the case of a general propagation direction, it is worth noting that the principal speeds  $v_{\mu||}$  and  $v_{\mu\perp}$  are stationary under first order variations of the propagation vector  $\nu^a$  around the eigenvector  $e_\mu^a$ . This is easily proved by taking the first order variation of the characteristic equation (108) induced by a first order variation  $\delta\nu^a$  in  $\nu^a$ ,

$$[\delta(v^2)(\rho h_{ac} + p_{ac}) - 2A_{abcd}\nu^b\delta\nu^d]\iota^c + [v^2(\rho h_{ac} + p_{ac}) - Q_{ac}]\delta\iota^c = 0. \quad (119)$$

This equation is supplemented by the restrictions

$$\nu_a\delta\nu^a = 0, \quad \iota_a\delta\iota^a = 0, \quad (120)$$

that follows from  $\nu^a$  and  $\iota^a$  being unit vectors. Requiring that  $\delta(v^2) = 0$  gives four linear equations for the three components of  $\delta\iota^a$ , namely eqs. (119) and (120). In order for the system not to be overdetermined these equations cannot be linearly independent. To prove that we do have linear dependence in the case when  $\nu^a$  is an eigenvector of  $k^a_b$  we contract eq. (119) with  $\iota^a$  which gives the condition

$$B_{bd}\nu^b\delta\nu^d = 0, \quad (121)$$

where  $B_{bd}$  is the symmetric tensor

$$B_{bd} = A_{abcd}\iota^a\iota^c = E_{abcd}\iota^a\iota^c - p_{bd}. \quad (122)$$

As we have seen, taking  $\nu^a$  to be an eigenvector of  $k^a_b$  leads to  $\iota^a$  being an eigenvector as well. In this case  $B_{bd}$  is diagonal in the eigenvector basis, leading to (121) being automatically satisfied. Thus, the number of independent equations for  $\delta\nu^a$  is at most three, which proves our statement that  $v$  is stationary under variations of propagation direction around a principal direction.

## 6 A particular equation of state

In this work will assume an equation of state where the energy per particle depends on only one invariant of  $\eta^a_b$ , namely the *shear scalar*

$$s^2 = \frac{1}{36} [(\eta^a_a)^3 - \eta^a_b\eta^b_c\eta^c_a - 24] = \frac{1}{2} \sum_{\mu=1}^3 s_\mu^2, \quad s_\mu^2 = \frac{1}{6} (z_\mu^{-1} - z_\mu)^2, \quad (123)$$

where, as before,  $z_\mu = n_{\mu+2}/n_{\mu+1}$ . This definition of  $s^2$  differs from that of Carter and Quintana, for reasons that are discussed in the appendix. As also discussed there, the equation of state is assumed to be of the form

$$\epsilon = \check{\epsilon} + \frac{\check{\mu}}{n} s^2, \quad (124)$$

where checked symbols denotes that the quantity is unsheared, *i.e.* depends on  $n$  only. The quantity  $\check{\epsilon}$  is the unsheared energy per particle that was discussed in section 2. The quantity  $\check{\mu}$ , on the other hand, is the *shear modulus* or *modulus of rigidity*. Equivalently, the equation of state can be given as

$$\rho = \check{\rho} + \sigma, \quad (125)$$

with  $\check{\rho} = n\check{\epsilon}$  being the *unsheared energy density* and  $\sigma = \check{\mu}s^2$  being the *shearing energy density*.

For this class of equations of states, the pressure tensor  $p_{ab}$  takes the form

$$p_{ab} = p h_{ab} + \pi_{ab}, \quad (126)$$

where

$$p = \check{p} + (\check{\Omega} - 1)\sigma, \quad \check{p} = n^2 \frac{d\check{\epsilon}}{dn}, \quad \check{\Omega} = \frac{n}{\check{\mu}} \frac{d\check{\mu}}{dn} \quad (127)$$

$$\pi_{ab} = \frac{1}{6} \check{\mu} [(\eta^c_c)^2 \eta_{<ab>} - \eta^{cd} \eta_{c<a} \eta_{b>d}]. \quad (128)$$

The quantity  $\check{p}$  will be referred to as the *unsheared pressure*. In the eigenvector basis the anisotropy pressure tensor  $\pi_{ab}$  is given by eq. (96), the eigenvalues being

$$\pi_\mu = \check{\mu}(\chi_{\mu+1} - \chi_{\mu+2}), \quad \chi_\nu = \frac{1}{6}(z_\nu^{-2} - z_\nu^2). \quad (129)$$

It is worth noting that the  $z_\mu$ -dependent quantities  $s_\mu^2$  and  $\chi_\mu$  satisfy

$$z_\mu \frac{\partial s_\mu^2}{\partial z_\mu} = -2\chi_\mu, \quad z_\mu \frac{\partial \chi_\mu}{\partial z_\mu} = -2(s_\mu^2 + \frac{1}{3}). \quad (130)$$

This facilitates the calculation of the principal compressibility moduli, which we find are given by

$$\beta_\mu = \beta + 4\sigma - 2\sigma_\mu + (2\check{\Omega} - 1)\pi_\mu, \quad (131)$$

where we have introduced the shorthand notation

$$\beta = n \frac{\partial p}{\partial n} = \check{\beta} + \frac{4}{3}\check{\mu} + \left[ (\check{\Omega} - 1)\check{\Omega} + \check{\beta} \frac{d\check{\Omega}}{d\check{p}} \right] \sigma. \quad (132)$$

Here, the quantity  $\check{\beta}$  is the *bulk modulus*

$$\check{\beta} = n \frac{d\check{p}}{dn} = (\check{\rho} + \check{p}) \frac{d\check{p}}{d\check{\rho}}, \quad (133)$$

and we have used  $n d\check{\Omega}/dn = \check{\beta} d\check{\Omega}/d\check{p}$ . Our identification of  $\check{\beta}$  and  $\check{\mu}$  as the bulk and shear moduli is justified in the appendix. As seen in section 5, the principal longitudinal speeds of sound are now given by

$$v_{\mu||}^2 = \frac{\beta_\mu}{\rho + p_\mu}. \quad (134)$$

On the other hand, the speeds of the principal transversal waves are straightforwardly calculated to be

$$v_{\mu\perp\nu}^2 = \frac{\check{\mu} + 2\sigma - \frac{1}{2}(\sigma_\mu + \sigma_\nu + \pi_\mu - \pi_\nu)}{\rho + p_\nu}. \quad (135)$$

Note that in the limit of zero shear,  $z_\mu \rightarrow 1$ , the principal speeds goes over into

$$v_{\mu||}^2 \rightarrow v_{||}^2 = \frac{\check{\beta} + \frac{4}{3}\check{\mu}}{\check{\rho} + \check{p}} = \frac{d\check{p}}{d\check{\rho}} + \frac{4}{3} \frac{\check{\mu}}{\check{\rho} + \check{p}} \quad (136)$$

$$v_{\mu\perp\nu}^2 \rightarrow v_{\perp}^2 = \frac{\check{\mu}}{\check{\rho} + \check{p}}, \quad (137)$$

in agreement with Carter's results [29].

## 7 The degenerate case of two equal eigenvalues and orthogonal eigenvectors

In this section we shall summarise some useful relations for the degenerate case when two of the eigenvalues of  $k^a_b$  are equal, *i.e.* when two of the linear particle densities, say  $n_2$  and  $n_3$ , are equal. This is the situation one for instance encounters in the spherically symmetric case. With this application in mind we will label quantities corresponding to the nondegenerate and degenerate eigenvalues by  $r$  (for radial) and  $t$  (for tangential) respectively. We thus set

$$n_r = n_1, \quad n_t = n_2 = n_3 \quad (138)$$

$$r_a = e_{1a}, \quad t_{ab} = h_{ab} - r_a r_b = e_{2a} e_{2b} + e_{3a} e_{3b}, \quad (139)$$

with  $t^a_b$  being the projection tensor that projects onto the degenerate eigenvector two-plane. The pulled back material space metric  $k_{ab}$  can now be written as

$$k_{ab} = n_r^2 r_a r_b + n_t^2 t_{ab}. \quad (140)$$

When calculating the pressure tensor we may take the view that we are dealing with an effective two-parameter equation of state  $\epsilon_{\text{eff}}(n_r, n_t) = \epsilon(n_r, n_t, n_t)$ . We are allowed to do so because the full three-parameter equation of state  $\epsilon(n_1, n_2, n_3)$  is invariant under permutations of the arguments, which in turn is due to the corresponding symmetry of the scalar invariants of  $k^a_b$ . Note however that, in general, we cannot use the effective equation of state when calculating higher order derivatives of  $\epsilon$ , since the first order derivatives do not obey the permutation symmetry. In particular, one needs the full three-parameter equation of state to calculate the elasticity tensor, which should not be surprising since it is used to determine the speed of wave propagation. Using the effective equation of state the pressure tensor may be written as

$$p_{ab} = p_r r_a r_b + p_t t_{ab}, \quad (141)$$

where

$$p_r = n n_r \frac{\partial \epsilon_{\text{eff}}}{\partial n_r}, \quad (142)$$

$$p_t = \frac{1}{2} n n_t \frac{\partial \epsilon_{\text{eff}}}{\partial n_t}. \quad (143)$$

For practical purposes it is often convenient to change the two effective equation of state parameters from  $(n_r, n_t)$  to  $(n, z)$ , with

$$n = n_r n_t^2, \quad z = n_r / n_t. \quad (144)$$

The function  $z$  represents the only independent quotient  $z_\mu = n_{\mu+1} / n_{\mu+2}$ , since clearly  $z_1 = 1$  and  $z_2^{-1} = z_3 = z$ . Making this change of parameters we find

$$k_{ab} = n^{2/3} \eta_{ab}, \quad \eta_{ab} = z^{1/3} (z r_a r_b + z^{-1} j_{ab}). \quad (145)$$

Defining a new traceless tensor

$$l_{ab} = -2 r_a r_b + j_{ab}, \quad (146)$$

we may calculate  $\pi_{ab}$  according to the very simple formula

$$\pi_{ab} = q l_{ab}, \quad (147)$$

where

$$q = \pi_t = -\frac{1}{2} \pi_r = -\frac{1}{2} n z \frac{\partial \epsilon_{\text{eff}}}{\partial z} \quad (148)$$

Thus, by specifying  $\epsilon = \epsilon_{\text{eff}}(n, z)$ , the pressure tensor is given by the relation

$$p_{ab} = p h_{ab} + q l_{ab}, \quad (149)$$

where  $p$ , as before, is the isotropic pressure

$$p = n^2 \frac{\partial \epsilon_{\text{eff}}}{\partial n} \quad (150)$$

As pointed out above, we cannot calculate all speeds of wave propagation using only the effective equation of state. However, for special cases of the directions of propagation and polarization, it is possible. Out of the nine speeds  $v_{\mu||}$  and  $v_{\mu\perp\nu}$  in the principal directions only five are independent, namely

$$\begin{aligned} v_{r||} &= v_{1||} \\ v_{t||} &= v_{2||} = v_{3||} \\ v_{r\perp} &= v_{1\perp 2} = v_{1\perp 3} \\ v_{t\perp r} &= v_{2\perp 1} = v_{3\perp 1} \\ v_{t\perp t} &= v_{2\perp 3} = v_{3\perp 2} \end{aligned} \quad (151)$$

Of these five only two actually requires the use of the full three-parameter equation of state, namely those for which both the propagation and polarization vectors lie in the degenerate two-plane, *i.e.*  $v_{t||}$  and  $v_{t\perp t}$ . The remaining three are given by

$$v_{r||}^2 = \frac{\beta_r}{\rho + p_r}, \quad \beta_r = n_r \frac{\partial p_r}{\partial n_r} \quad (152)$$

$$v_{r\perp}^2 = \frac{n_t^2(p_t - p_r)}{n_t^2 - n_r^2} = \frac{3q}{1 - z^2} \quad (153)$$

$$v_{t\perp r}^2 = \frac{n_r^2(p_r - p_t)}{n_r^2 - n_t^2} = \frac{3q}{z^{-2} - 1}. \quad (154)$$

Having the full three-parameter equation of state of course makes it unnecessary to be cautious about setting  $n_2 = n_3$  before calculating various quantities, since we may simply specialise the general formulae to the degenerate case, *i.e.* set (or take the limit)  $n_2 = n_3$  afterwards. Doing this for the equation of state treated in section 6, we find

$$s_r^2 = s_1^2 = 0 \\ s_t^2 = s_2^2 = s_3^2 = \frac{1}{6}(z^{-1} - z)^2. \quad (155)$$

$$s^2 = \frac{1}{2} \sum_{\mu=1}^3 s_\mu^2 = s_t^2. \quad (156)$$

With  $\chi_1 = 0$  identically vanishing, we denote the only independent  $\chi_\mu$  by  $\chi$  (without a basis index),

$$\chi = -\chi_2 = \chi_3 = \frac{1}{6}(z^{-2} - z^2). \quad (157)$$

The anisotropy pressure scalar  $q$  is now simply given by

$$q = \check{\mu}\chi. \quad (158)$$

To conclude this section we give the principal speeds for this equation of state,

$$v_{r||}^2 = \frac{\beta + 4[\sigma - (\check{\Omega} - \frac{1}{2})q]}{\rho + p_r} \\ v_{t||}^2 = \frac{\beta + 2[\sigma + (\check{\Omega} - \frac{1}{2})q]}{\rho + p_t} \\ v_{r\perp}^2 = \frac{\check{\mu} + \frac{3}{2}(\sigma + q)}{\rho + p_t} \\ v_{t\perp r}^2 = \frac{\check{\mu} + \frac{3}{2}(\sigma - q)}{\rho + p_r} \\ v_{t\perp t}^2 = \frac{\check{\mu} + \sigma}{\rho + p_t}, \quad (159)$$

where, as in the general case,

$$\beta = \check{\beta} + \frac{4}{3}\check{\mu} + \left[ (\check{\Omega} - 1)\check{\Omega} + \check{\beta} \frac{d\check{\Omega}}{d\check{p}} \right] \sigma \quad (160)$$

## 8 Application to static spherically symmetric configurations

In this section we shall apply the elasticity theory just described to study static spherically symmetric configurations, with neutron stars as the obvious astrophysical objects in mind. It is remarkable that

although elastic spherically symmetric configurations have been studied in the past [26, 27], it seems that not a single solution has been published, whether exact or numerical, that can be considered as a rough model for a neutron star. Referring to section 7 and using Schwarzschild coordinates, we write the spacetime metric as

$$g_{ab} = -u_a u_b + r_a r_b + t_{ab}, \quad (161)$$

where

$$u_a = -e^\nu (dt)_a \quad (162)$$

$$r_a = e^\lambda (dr)_a, \quad e^{-2\lambda} = 1 - \frac{2m}{r} \quad (163)$$

$$t_{ab} = r^2 (d\Omega^2)_{ab}, \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (164)$$

Generically, the stress-energy tensor will only be compatible with a spherically symmetric spacetime if the material space metric  $k_{AB}$  is also spherically symmetric and, in addition, the  $\text{SO}(3)$  symmetry orbits of  $M'$  and  $X'$  are identified by the material space mapping  $\psi$ . Hence, we set

$$k_{AB} = \tilde{r}_A \tilde{r}_B + \tilde{t}_{AB}, \quad (165)$$

where, analogously to the above,

$$\tilde{r}_A = e^{\tilde{\lambda}} (d\tilde{r})_A \quad (166)$$

$$\tilde{t}_{AB} = \tilde{r}^2 (d\tilde{\Omega}^2)_{AB}, \quad d\tilde{\Omega}^2 = d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2. \quad (167)$$

The mapping  $\psi$  is now, up to irrelevant  $\text{SO}(3)$  transformations, defined through

$$\tilde{r} = \tilde{r}(r), \quad d\tilde{\Omega}^2 = d\Omega^2. \quad (168)$$

The pulled back material tensor  $k_{ab}$  is given by eq. (140) with

$$n_r = e^{\tilde{\lambda}-\lambda} \frac{d\tilde{r}}{dr}, \quad n_t = \frac{\tilde{r}}{r}. \quad (169)$$

It is common to assume a flat material space metric, *i.e.* to set  $\tilde{\lambda} = 0$ . However, we will keep  $\tilde{\lambda}$  unspecified for the time being as other choices may be of interest as well [36]. The particle density can be written

$$n = n_r n_t^2 = (\tilde{r}/r)^3 z \quad (170)$$

where, as in section 7,  $z$  is the dimensionless quotient

$$z = \frac{n_r}{n_t} = e^{\tilde{\lambda}-\lambda} \frac{r}{\tilde{r}} \frac{d\tilde{r}}{dr} \quad (171)$$

Regardless of the matter sources, the Einstein equations for any static spherically symmetric spacetime can be formulated as (*cf.* [37])

$$\frac{d\nu}{dr} = \frac{m + \frac{1}{2}\kappa r^3 p_r}{r(r-2m)} \quad (172)$$

$$\frac{dm}{dr} = \frac{1}{2}\kappa r^2 \rho \quad (173)$$

$$\frac{dp_r}{dr} = -(p_r + \rho) \frac{m + \frac{1}{2}\kappa r^3 p_r}{r(r-2m)} + \frac{6q}{r}, \quad q = \frac{1}{3}(p_t - p_r) \quad (174)$$

However, unless a one-parameter equation of state gives  $\rho$  and  $p_t$  as known functions of  $p_r$ , this system is underdetermined. In our case we are, as discussed in section 7, effectively dealing with a two-parameter

equation of state. An additional equation of motion can readily be found if we, to begin with, use  $n_r$  and  $n_t$  as the two independent matter variables. Indeed, eq. (169) directly implies that the evolution equation for  $n_t$  is

$$\frac{dn_t}{dr} = \frac{1}{r} \left( e^{\lambda-\bar{\lambda}} n_r - n_t \right). \quad (175)$$

An evolution equation for  $n_r$  can now be found by using

$$\frac{dp_r}{dr} = \frac{\partial p_r}{\partial n_r} \frac{dn_r}{dr} + \frac{\partial p_r}{\partial n_t} \frac{dn_t}{dr}, \quad (176)$$

which we solve for  $dn_r/dr$ , yielding

$$\frac{dn_r}{dr} = \frac{n_r}{\beta_r} \left( \frac{dp_r}{dr} - \frac{\partial p_r}{\partial n_t} \frac{dn_t}{dr} \right), \quad (177)$$

with  $dp_r/dr$  and  $dn_t/dr$  given by eq. (174) and eq. (175), respectively. Knowing  $\rho$ ,  $p_r$  and  $q$  as functions of  $n_r$  and  $n_t$ , eqs. (173), (175) and (177) provides a closed system of three equations for three unknown variables  $m$ ,  $n_r$  and  $n_t$ . The gravitational potential  $\nu$  is decoupled from this system and hence its governing equation (172) can be integrated afterwards.

From a physical point of view, it is more natural to give the equation of state in terms of  $n$  and  $z$  rather than in terms of  $n_r$  and  $n_t$ . To transform eqs. (175) and (177) into evolution equations for  $n$  and  $z$ , we begin by noting that  $n_t^3 = n/z$  leads to

$$\frac{dz}{dr} = z \left( \frac{1}{n} \frac{dn}{dr} - \frac{3}{n_t} \frac{dn_t}{dr} \right), \quad (178)$$

where  $n_t^{-1} dn_t/dr$  can be reexpressed as

$$\frac{1}{n_t} \frac{dn_t}{dr} = \frac{1}{r} \left( e^{\lambda-\bar{\lambda}} z - 1 \right). \quad (179)$$

Moreover, by viewing  $p_r$  as a function of  $(n, z)$  and using eq. (178), we find

$$\frac{dp_r}{dr} = \frac{\beta_r}{n} \frac{dn}{dr} - \frac{3}{n_t} \frac{dn_t}{dr} z \frac{\partial p_r}{\partial z}, \quad (180)$$

which can be solved for  $dn/dr$ , resulting in

$$\frac{dn}{dr} = \frac{n}{r\beta_r} \left[ -(\rho + p_r) \frac{m + \frac{1}{2}\kappa r^3 p_r}{r - 2m} + 6q + 3z \frac{\partial p_r}{\partial z} \left( e^{\lambda-\bar{\lambda}} z - 1 \right) \right], \quad (181)$$

where we have used eqs. (174) and (179). Inserting eq. (179) into eq. (178) as well, the evolution equation for  $z$  is

$$\frac{dz}{dr} = z \left[ \frac{1}{n} \frac{dn}{dr} - \frac{3}{r} \left( e^{\lambda-\bar{\lambda}} z - 1 \right) \right], \quad (182)$$

where  $dn/dr$  is to be inserted from eq. (181). Summarising, in the variables  $(m, n, z)$ , Einstein's equations take the form

$$\frac{dm}{dr} = \frac{1}{2}\kappa r^2 \rho \quad (183)$$

$$\frac{dn}{dr} = \frac{n}{r\beta_r} \left[ -(\rho + p_r) \frac{m + \frac{1}{2}\kappa r^3 p_r}{r - 2m} + 6q + 3z \frac{\partial p_r}{\partial z} \left( e^{\lambda-\bar{\lambda}} z - 1 \right) \right] \quad (184)$$

$$\frac{dz}{dr} = z \left[ \frac{1}{n} \frac{dn}{dr} - \frac{3}{r} \left( e^{\lambda-\bar{\lambda}} z - 1 \right) \right], \quad (185)$$

where  $\rho$ ,  $p_r$ ,  $q$  and  $\beta_r$  are related by a two-parameter equation of state, given by  $\epsilon = \epsilon_{\text{eff}}(n, z)$  and

$$\rho = n\epsilon, \quad p_r = n^2 \frac{\partial \epsilon}{\partial n} - 2q, \quad q = -\frac{1}{2}nz \frac{\partial \epsilon}{\partial z}, \quad \beta_r = n \frac{\partial p_r}{\partial n} + z \frac{\partial p_r}{\partial z}. \quad (186)$$

We shall now specialise the system of equations (183) - (185) to the particular equation of state introduced in section 6. To facilitate a comparison with the perfect fluid case, we use the relation  $dn/n = d\check{p}/\check{\beta}$  to replace the particle density  $n$  with the unsheared pressure  $\check{p}$  as one of the independent variables. We find that, with  $(m, \check{p}, z)$  as independent variables, Einstein's equations can be put in the form

$$\frac{dm}{dr} = \frac{1}{2}\kappa r^2 \rho \quad (187)$$

$$\frac{d\check{p}}{dr} = \frac{\check{\beta}}{r\beta_r} \left\{ -(\rho + p_r) \frac{m + \frac{1}{2}\kappa r^3 p_r}{r - 2m} + 6q + 4 \left( e^{\lambda - \tilde{\lambda}} z - 1 \right) [\check{\mu} + 3\sigma + \frac{3}{2}(1 - \check{\Omega})q] \right\} \quad (188)$$

$$\frac{dz}{dr} = \frac{z}{r} \left[ \frac{r}{\check{\beta}} \frac{d\check{p}}{dr} - 3 \left( e^{\lambda - \tilde{\lambda}} z - 1 \right) \right], \quad (189)$$

where

$$\sigma = \check{\mu} s^2, \quad s^2 = \frac{1}{6}(z^{-1} - z)^2, \quad (190)$$

$$q = \check{\mu} \chi, \quad \chi = \frac{1}{6}(z^{-2} - z^2), \quad (191)$$

$$\rho = \check{\rho} + \sigma, \quad (192)$$

$$p_r = p - 2q, \quad p = \check{p} + (\check{\Omega} - 1)\sigma, \quad (193)$$

$$\check{\beta} = (\check{\rho} + \check{p}) \frac{d\check{p}}{d\check{\rho}}, \quad (194)$$

$$\beta_r = \beta + 4 \left[ \sigma + (\check{\Omega} - \frac{1}{2})q \right], \quad \beta = \check{\beta} + \frac{4}{3}\check{\mu} + \left[ \check{\Omega}(\check{\Omega} - 1) + \check{\beta} \frac{d\check{\Omega}}{d\check{p}} \right] \sigma, \quad (195)$$

$$\check{\Omega} = \frac{\check{\beta} d\check{\mu}}{\check{\mu} d\check{p}}. \quad (196)$$

and where  $\check{\rho}$  and  $\check{\mu}$  are related to  $\check{p}$  through the equation of state. Note that, for vanishing shear modulus  $\check{\mu}$ , the equation for  $z$  is decoupled and the remaining system reduces to the standard perfect fluid form. This makes this explicitly closed system useful for comparing elastic models with the corresponding perfect fluid ones.

It is interesting to evaluate the derivative  $dz/dr$  at zero shear  $z = 1$ . Using the fact that  $\sigma = q = 0$  at such a point we readily find

$$\left. \frac{dz}{dr} \right|_{z=1} = -\frac{1}{rv_{||}^2} \left[ \frac{m + \frac{1}{2}\kappa r^3 \check{p}}{r - 2m} + 3 \frac{d\check{p}}{d\check{\rho}} \left( e^{\lambda - \tilde{\lambda}} - 1 \right) \right] \quad (197)$$

Using a flat material space metric, *i.e.* setting  $\tilde{\lambda} = 0$ , the right hand side is clearly negative for any reasonable equation of state, since  $\lambda$  is then everywhere positive[38]. Therefore, unless one chooses  $z > 1$  at a phase transition, see below,  $z$  will always remain less than or equal to unity. Hence the radial pressure can never exceed the tangential pressure unless one chooses it to do so at the phase transition. One may suspect that such a choice of phase transition would tend to make the star radially unstable, but this will be the subject for later investigation[39].

## 9 Boundary conditions

There are three different types of boundaries that one might consider; the center of the star  $r = 0$ , a transition between different materials in the star and the surface of the star  $r = R$ . At the center we require

that the spacetime is regular (elementary flat). This implies pressure isotropy  $p_{ab} = p h_{ab}$  and, given this, the conditions are the same as for a perfect fluid, namely finiteness of the energy density  $\rho$  and pressure  $p$  as well as vanishing of the mass function  $m$ . Since the pressure isotropy condition is fulfilled iff  $q = 0$ , eq. (158) shows that unless  $\tilde{\mu} = 0$ , the function  $z$  must take its unsheared value 1, meaning that the shear  $s^2$  must vanish. In both cases  $\tilde{p} = p_r = p_t = p$  and  $\rho = \tilde{\rho}$  hold at the center. Since the right hand sides of the evolution equations are all zero at  $r = 0$  we are forced to move out the starting point of integration a small distance from the center when solving the evolution equations numerically. The initial conditions for the integration is thus given by the following expansions around  $r = 0$ :

$$m = \frac{1}{6} \kappa \tilde{\rho}_c r^3 + \dots \quad (198)$$

$$\tilde{p} = \tilde{p}_c - \kappa \tilde{\beta}_c \frac{(\tilde{\rho}_c + \tilde{p}_c)(\tilde{\rho}_c + 3\tilde{p}_c) - 4\tilde{\mu}_c \tilde{\rho}_c}{12(\tilde{\beta}_c + \frac{4}{3}\tilde{\mu}_c)} r^2 + \dots \quad (199)$$

$$z = 1 - \kappa \frac{(\tilde{\rho}_c + \tilde{p}_c)(\tilde{\rho}_c + 3\tilde{p}_c) + 3\tilde{\beta}_c \tilde{\rho}_c}{30(\tilde{\beta}_c + \frac{4}{3}\tilde{\mu}_c)} r^2 + \dots \quad (200)$$

where a subscript  $c$  denotes evaluation at  $r = 0$ .

The remaining types of boundary conditions are governed by the continuity of the first and second fundamental forms[40], which for all static spherically symmetric spacetimes leads to continuity of the Schwarzschild radial coordinate  $r$ , the mass function  $m$  (no surface layers), the gravitational potential  $\nu$  and its first derivative  $d\nu/dr$ . Via eq. (172), the continuity of  $d\nu/dr$  can be replaced by continuity of the radial pressure  $p_r$ . Since we are using  $r$  as our radial variable, its continuity requirement is trivially imposed and will not be discussed further.

At the surface of the star where we match the interior solution to a Schwarzschild vacuum solution of mass  $M$  the boundary conditions reduce to

$$m|_{r=R} = M \quad (201)$$

$$\nu|_{r=R} = \frac{1}{2} \ln \left( 1 - \frac{2M}{R} \right) \quad (202)$$

$$p_r|_{r=R} = 0. \quad (203)$$

An interesting observation is that, for  $\tilde{\mu} = k\tilde{p}$  with  $k$  constant, eqs. (203) and (193) imply

$$z^2|_{r=R} = \frac{-3 - (1 - \tilde{\Omega}_s)k + \sqrt{9 + 6(1 - \tilde{\Omega}_s)k + 4k^2}}{k(1 + \tilde{\Omega}_s)} \quad (204)$$

where  $\tilde{\Omega}_s = \tilde{\Omega}|_{r=R}$ . Remarkably, if the isotropic part of the equation of state is a relativistic polytrope, in which case  $\tilde{\Omega}$  is constant (and actually equal to the adiabatic index, see section 10), it follows that the value of  $z$  at the surface is known *a priori* in terms of the equation of state parameters  $\tilde{\Omega}$  and  $k$ .

Turning now to an interior boundary between two different phases of matter, the continuity of  $m$  and  $\nu$  can be imposed without difficulties. On the other hand the continuity of the radial pressure  $p_r$  requires some thought since we are using the unsheared pressure  $\tilde{p}$ , rather than  $p_r$ , as an unknown variable. Since the junction conditions have nothing directly to say about the shear variable  $z$ , there is in fact no unique way of making the matching. This means that the uniqueness result of Park[27] does not generalise to the case of elastic models with transitions between different phases of matter; if phase transitions are allowed for there is *not* a unique static spherically symmetric model for a given value of the central pressure  $p_c$ . Since a real neutron star hardly can be viewed as one big piece of elastic matter, this complication is not of pure academic nature. Although a particular class of equations of state is considered here, it should be obvious that non-uniqueness is not peculiar to this class. However, we shall see below that the case when the outer matter phase is a perfect fluid is an important exception for which uniqueness of the matching does apply.

Now, according to eq. (193), we are to match the following combination of the unknown variables  $\tilde{p}$  and  $z$ :

$$\tilde{p} - \frac{\tilde{\mu}}{6} [(1 - \tilde{\Omega})(z^{-1} - z)^2 + 2(z^{-2} - z^2)], \quad (205)$$



where it is to be recalled that  $\check{\mu}$  and  $\check{\Omega}$  are in general functions of  $\check{p}$ . For simplicity as well as for all practical purposes of this paper, we shall restrict ourselves to the situation when one of the phases is a perfect fluid ( $\check{\mu}$  identically vanishing) while the other is a general elastic phase. Since the equations are always integrated from the center and outwards, the values  $\check{p}_-$  and  $z_-$  of  $\check{p}$  and  $z$  on the interior side of the boundary will be given while the exterior values  $\check{p}_+$  and  $z_+$  are to be determined. In fact, this implies that the cases when the elastic phase is interior to the perfect fluid phase and vice versa need to be treated separately. Before we do so, we note that for the perfect fluid phase, the value of  $z$  is completely irrelevant so in fact it is meaningless to even define  $z$  in such a region of the star. Now, starting with an elastic phase interior to a perfect fluid, the matching condition becomes

$$\check{p}_+ = \check{p}_- - \frac{\check{\mu}_-}{6} [(1 - \check{\Omega}_-)(z_-^{-1} - z_-)^2 + 2(z_-^{-2} - z_-^2)], \quad (206)$$

which uniquely determines  $\check{p}_+$ , while, as we just concluded,  $z_+$  need not even be defined. If, on the other hand, the interior phase is the perfect fluid, the junction condition is

$$\check{p}_+ - \frac{\check{\mu}_+}{6} [(1 - \check{\Omega}_+)(z_+^{-1} - z_+)^2 + 2(z_+^{-2} - z_+^2)] = \check{p}_-. \quad (207)$$

This leads to a one parameter family of possible phase transitions characterised for instance by the value of  $z_+$ . Setting  $z_+ = 1$ , it follows that  $\check{p}$  as well as all other pressures are continuous across the boundary.<sup>1</sup> This is clearly the simplest case to consider. It is interesting to observe that although  $z_+ = 1$  implies continuity of  $\check{p}$  the converse is not necessarily true. Namely, setting  $\check{p}_+ = \check{p}_-$  leads to two solutions for  $z_+^2$ ,

$$z_+^2 = 1 \quad \text{or} \quad z_+^2 = -\frac{3 - \check{\Omega}_+}{1 + \check{\Omega}_+}, \quad (208)$$

where the second equality can only be valid if  $\check{\Omega}_+ > 3$ . The second solution does not imply continuity of the other pressures but is probably unphysical due to the large value of  $\check{\Omega}_+$ .

## 10 Example: relativistic polytropes

In order to concretise the theory developed so far, we shall here consider a few examples of static spherically symmetric configurations. The first step is to specify the equation of state in terms of two functions of state *e.g.*  $\check{\rho}(\check{p})$  and  $\check{\mu}(\check{p})$ . For simplicity we shall only consider models where the one-parameter relation between  $\check{\rho}$  and  $\check{p}$  is that of a relativistic polytrope, *i.e.*

$$\check{\rho} = \frac{p_1}{\check{\Gamma} - 1} \left[ \frac{\check{p}}{p_1} + \left( \frac{\check{p}}{p_1} \right)^{1/\check{\Gamma}} \right], \quad (209)$$

where  $\check{\Gamma} = \check{\beta}/\check{p}$  is the (constant) *adiabatic* or *compressibility index* and  $p_1$  is the pressure which signals the transition between the classical polytropic  $\check{\rho} \propto \check{p}^{\check{\Gamma}}$  and the linear  $\check{\rho} \propto \check{p}$  regime. More precisely, at  $\check{p} = p_1$ , the polytropic part and the linear part contributes equally to  $\check{\rho}$ . This unsheared part of the equation of state follows from

$$\check{\epsilon} = \frac{p_1 v_0}{\check{\Gamma} - 1} \left[ (v_0 n)^{\check{\Gamma}-1} + 1 \right] \quad (210)$$

where  $v_0$  is some constant with the dimension of volume. The shear modulus  $\check{\mu}$  will be assumed to be piecewise proportional to the unsheared pressure,  $\check{\mu} = k\check{p}$ , where  $k$  is a dimensionless constant which is allowed to vanish. This assumption is supported by several calculations regarding the equation of state of neutron stars, see *e.g.* [41] for a review. For this class of equations of state the compressibility index  $\check{\Gamma}$  will be constant throughout the star. Moreover  $\check{\Omega}$  will be piecewise constant and equal to  $\check{\Gamma}$  where it is non-zero. We chose the constant to be  $\check{\Gamma} = 5/3$  corresponding to a non-relativistic Fermi gas. The transition pressure

<sup>1</sup>Note however that the total energy density  $\rho$  may still be discontinuous, but only if  $\check{\rho}$  is.

$p_1$  is chosen to be  $1.5 \times 10^{37}$  dyn/cm<sup>2</sup> in order to give total radii in the order of 10 km. We shall also use the simplifying assumption that the material space metric is flat, *i.e.* we choose  $\bar{\lambda} = 0$ .

We consider first the situation where we have a rigid crust afloat on a perfect fluid core. The transition from the fluid to the solid phase is taken to be approximately at the neutron drip pressure,  $\tilde{p} = 6 \times 10^{29}$  dyn/cm<sup>2</sup>, and we let  $k$  take values in the range  $0 \leq k \leq 0.1$ . Of course a zero value corresponds to a perfect fluid star and therefore gives a good reference model. The highest value 0.1 is probably unphysically large but is included in order to give insights to extreme cases. The phase transition is chosen such that all pressures are continuous across the matching surface. The first question we ask is how the maximum mass and stability properties change when we include the effect of a crust. For a few models with different  $k$  we have plotted the total mass  $M$  as a function of the total radius  $R$ . This allows us to read off the maximum mass attainable for these models. Moreover, at least in the case of a perfect fluid, an extremum of this curve correspond to a change in the stability for one radial normal mode[42]. Whether or not this is also the case when the matter is allowed to behave elastically is currently being investigated[39], but we anticipate that it will due to the following argument. For well-behaved equations of state the problem of calculating time-harmonic radial perturbations turn into a Sturm-Liouville problem with the squared frequency  $\omega^2$  as the eigenvalues. Since the eigenvalues of such a problem can be ordered, with the lowest eigenvalue  $\omega_0^2$  corresponding to an eigenfunction without nodes, it is clear that instability sets in at points where  $\omega_0 = 0$ . Allowing only perturbations which conserves the total mass of the star it is seen that only stationary points in the mass-radius diagram correspond to zero frequency modes, since it is only there the perturbed star is still in the equilibrium sequence, and hence does not oscillate. We point out however that if the equation of state involves discontinuities (as, for instance, is the case for our choice of shear modulus) one has to be careful since the perturbation problem will in general no longer be of the strict Sturm-Liouville type. For the types of discontinuities considered in this paper we do not expect that the general conclusion will change.

In figure 2 we see, as expected, that adding a solid crust does not affect the maximum mass significantly, even for very large  $k$ . However, for stars with large total radii, corresponding to low central pressures, the curves differ significantly. One should however be careful when interpreting this diagram since each model should be viewed as parametrised by the central pressure which is not apparent in fig. 2. For this reason it is useful to plot also the *compactness*  $M/R$  as a function of the central pressure, as done in fig. 3. Here it is clear that the compactness does not change significantly with increasing shear modulus, even for low central pressures. However, the obvious trend is that the compactness increases with increasing shear modulus.

We turn, now, to the case when we instead have a solid matter phase in the core. We take the transition pressure to be  $2 \times 10^{34}$  dyn/cm<sup>2</sup>. In this case the proportionality constant  $k$  is expected to be smaller than in the crust[41], if it is non-zero at all. We therefore choose to look at the interval  $0 \leq k \leq 0.02$ , where again the largest value is expected to be unphysical. We again look at the mass-radius diagram (figure 4) and see that the maximum mass actually *decreases* with increasing  $k$  in this case. However, looking at the compactness we note that the maximum value increases. This can be understood from the mass-radius diagram since the compactness can be parametrised by the angle between a ray from the origin to  $(r, m)$  and the  $r$ -axis. Clearly the maximum compactness is reached after the peak of the mass-radius curve, corresponding to an unstable model. One interesting possibility for neutron stars is that they, if compact enough, could allow trapped  $w$ -modes[43]. Based on perfect fluid calculations however, it does not seem likely that this is the case for real neutron stars[44]. Speculatively, adding a rigid core could provide the extra compactness needed. As discussed by Rosquist[45], it is not the total compactness that is important but rather some partial measure of compactness that determines if a stellar configuration allow for trapped  $w$ -modes. A necessary (but not sufficient) condition is that  $\nu'r = 1$  somewhere in the star. The models here does not allow for trapped  $w$ -modes, but a higher  $k$  in the core does indeed pushes  $\nu'r$  towards one.

In order to assess the impact of the choice of parameters we have calculated a large number of models with various parameter values. The effects of a non-zero shear modulus do not appear to depend strongly on the choice of  $p_1$  or  $p_t$ . Also, adding even rather large discontinuities in the tangential pressure in the fluid-solid phase transition does not seem to have any major impact either.

## 11 Comments and outlook

As far as our study of static spherically symmetric solutions to the Einstein equations with elastic matter source is concerned, we see it as a first foundation laying step towards a study of more generic configurations. The obtained numerical solutions may not be highly interesting in themselves, as they, at least for realistic choices of the shear modulus  $\tilde{\mu}$ , do not differ drastically from the corresponding perfect fluid model. However, it is important to have a good control and understanding of the background model when perturbing a spherically symmetric solution into less symmetric neighbours. A feature of the presented models which we see as oversimplified is that the equation of state is set up using a single flat material space metric throughout a large region of the star, stretching over several orders of magnitude in energy density and pressure. While the usage of a flat material space metric is naturally suggested from first principles (see the end of section 3) one should not expect the stellar material to be ordered over such long ranges as being considered here, which means that the use of a flat metric as the single reference structure is not well justified. Instead, the crust of a neutron star is layered where each layer has a different nuclear composition and lattice structure[41]. Thus, a proper neutron star model should include several phases of matter, each well, at least effectively, described by a flat material space metric. We believe that our formalism is well suited to deal with these kinds of problems as well as with perturbation calculations of various kinds.

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## Appendix A: Expansion around a locally unsheared state

To motivate our choice of invariant as well as our equation of state it is instructive to make an expansion of the energy around a locally unsheared state, *i.e.* a state where  $h_{ab} = n_0^{-2/3} k_{ab}$  for some particle density  $n = n_0$ . Using the notation  ${}^0\eta_{ab} = \eta_{ab}|_{n=n_0} = n_0^{-2/3} k_{ab}$ , we introduce the *strain tensor*  $e_{ab}$  as

$$e_{ab} = \frac{1}{2}(h_{ab} - {}^0\eta_{ab}). \quad (211)$$

In the following, we shall work exclusively with tensors having mixed indices and formulate everything as (three-dimensional) matrix equations in index free notation. Thus needing to raise one of the indices of  $e_{ab}$ , we have two natural choices to do so, either by using  $g^{ab}$  or  ${}^0\eta^{-1ab} = n_0^{2/3} k^{-1ab}$ . We shall for the moment work with both, as the former is more natural from a spacetime point of view, while the latter connects more closely to the classical theory and is the one used by Carter and Quintana.

Now, with these conventions we introduce  $\mathbf{e}$  and  $\hat{\mathbf{e}}$  to be the matrices with indices

$$e^a{}_b = \frac{1}{2}(h^a{}_b - {}^0\eta^a{}_b), \quad \hat{e}^a{}_b = \frac{1}{2}({}^0\eta^{-1a}{}_b - h^a{}_b), \quad (212)$$

or in index-free form

$$\mathbf{e} = \frac{1}{2}(\mathbb{1} - {}^0\boldsymbol{\eta}), \quad \hat{\mathbf{e}} = \frac{1}{2}({}^0\boldsymbol{\eta}^{-1} - \mathbb{1}). \quad (213)$$

Note that, formally,  $\mathbf{e}$  and  $\hat{\mathbf{e}}$  are related by

$$\hat{\mathbf{e}} = \frac{\mathbf{e}}{\mathbb{1} - 2\mathbf{e}}, \quad \mathbf{e} = \frac{\hat{\mathbf{e}}}{\mathbb{1} + 2\hat{\mathbf{e}}}. \quad (214)$$

To proceed we expand the particle density to second order in  $\mathbf{e}$  and  $\hat{\mathbf{e}}$  respectively,

$$\frac{n}{n_0} \simeq 1 - \text{Tr } \mathbf{e} + \frac{1}{2}(\text{Tr } \mathbf{e})^2 - \text{Tr } \mathbf{e}^2 \quad (215)$$

$$\simeq 1 - \text{Tr } \hat{\mathbf{e}} + \frac{1}{2}(\text{Tr } \hat{\mathbf{e}})^2 + \text{Tr } \hat{\mathbf{e}}^2, \quad (216)$$

where “ $\simeq$ ” means “equal up to and including quadratic terms in  $\mathbf{e}$ ”. Note that the only formal difference between these two expressions is the plus sign in front of  $\text{Tr } \hat{\mathbf{e}}^2$ . So, since any quantity expanded to second order in  $\mathbf{e}$  will be a linear function of  $\text{Tr } \mathbf{e}$ ,  $\text{Tr } \mathbf{e}^2$  and  $(\text{Tr } \mathbf{e})^2$ , it follows that such an expansion will look formally the same when instead expressed in  $\hat{\mathbf{e}}$ , if we use eq. (215) to replace all occurrences of  $\text{Tr } \hat{\mathbf{e}}$  in favour of  $n/n_0 - 1$ . In the following we will therefore use only  $\mathbf{e}$ . Now, in the classical theory of elasticity, one usually relates thermodynamical quantities to unit volume of the completely relaxed reference state. Hence, if we as our reference state take the unsheared state  $\mathbf{e} = 0$ , which however is not necessarily completely relaxed, it is  $E = n_0\epsilon$  rather than  $\rho = n\epsilon$  that should be compared to the energy per unit volume of the classical works. Without assuming anything about the equation of state we write the expansion as

$$E \simeq E_0 + p_0 \left( \frac{n}{n_0} - 1 \right) + \frac{1}{2} \lambda_0 (\text{Tr } \mathbf{e})^2 + \mu_0 \text{Tr}(\mathbf{e}^2), \quad (217)$$

where  $E_0$ ,  $p_0$ ,  $\lambda_0$  and  $\mu_0$  are the expansion coefficients. In general, these coefficients depend on the choice of  $n_0$ . It is apparent from eq. (217) that the unsheared state can only be a completely relaxed state, *i.e.* a state that minimises  $E$  and hence also  $\epsilon$ , if

$$p_0 = 0, \quad \lambda_0 \geq 0, \quad \mu_0 \geq 0. \quad (218)$$

In this case we may identify the constants  $\lambda_0$  and  $\mu_0$  as the *Lamé coefficients*. The constant  $\mu_0$  is also called the *shear modulus*, while the combination  $\beta_0 = \lambda_0 + \frac{2}{3}\mu_0$  is called the *bulk modulus*. However, we warn the reader that this definition of the bulk modulus does not generalise to the case  $p_0 \neq 0$ , a point which we will return to later in this appendix.

We may find an interpretation for  $p_0$  if we note that for small  $\mathbf{e}$  the linear term dominates over the quadratic and higher order terms and we have approximately the thermodynamic relation

$$\Delta E \approx -p_0 \Delta V, \quad (219)$$

where  $\Delta E = E - E_0$ ,  $\Delta V = -(n - n_0)/n$  with  $\Delta V$  being the change in the volume. Thus it follows that  $p_0$  may be identified as pressure. In situations with small pressure, for instance near the surface of a star, it would be sufficient to use a *Hookean* equation of state *i.e.* to let the equation of state be given by eq. (217) with the cubic and higher order terms simply dropped and with the conditions (218) satisfied. However, since we are interested in the interior of neutron stars where pressures can be extreme, a more general approach is called for. As noted previously in section 2 we want the scalar invariants of  $\mathbf{k}$  used to set up the equation of state to consist of the particle density  $n$  along with up to two scalar invariants of the unit determinant matrix  $\boldsymbol{\eta}$ . As the invariants  $\text{Tr } \mathbf{e}$  and  $\text{Tr}(\mathbf{e}^2)$  of  $\mathbf{e}$  used in the above expansions are not exact invariants of  $\boldsymbol{\eta}$ , we need to reformulate things a little bit. So, following CQ we introduce the *constant volume shear tensor*  $s_{ab}$

$$s_{ab} = \frac{1}{2}(h_{ab} - \eta_{ab}), \quad (220)$$

which in contrast to the shear tensor  $e_{ab}$  does not depend on the particle density  $n_0$  of the reference state. As was the case with the shear tensor, there are two natural ways to raise one index, namely by using either  $h^{ab}$  or  $\eta^{-1ab}$ , giving us the matrices

$$\mathbf{s} = \frac{1}{2}(\mathbb{1} - \boldsymbol{\eta}), \quad \hat{\mathbf{s}} = \frac{1}{2}(\boldsymbol{\eta}^{-1} - \mathbb{1}) \quad (221)$$

Expanding an arbitrary integer power of  $\boldsymbol{\eta}$  to second order in  $\mathbf{e}$  gives

$$\text{Tr}(\boldsymbol{\eta}^n) \simeq 3 + 2n^2 s^2, \quad n \in \mathbb{Z}, \quad (222)$$

where

$$s^2 \simeq \text{Tr}(\mathbf{e}^2) - \frac{1}{3}(\text{Tr } \mathbf{e})^2 = \text{Tr} \left\{ \left[ \mathbf{e} - \frac{1}{3}(\text{Tr } \mathbf{e})\mathbb{1} \right]^2 \right\}. \quad (223)$$

Note that  $\boldsymbol{\eta}$  has only one independent invariant up to second order in  $\mathbf{e}$ , which therefore also will be the case for  $\mathbf{s}$ . Namely, it follows that

$$\text{Tr } \mathbf{s} \simeq \text{Tr } \mathbf{s}^2 \simeq s^2 \simeq \text{Tr } \hat{\mathbf{s}} \simeq \text{Tr } \hat{\mathbf{s}}^2. \quad (224)$$

It is now instructive to reexpress the energy  $E$  as a function of  $n/n_0 - 1$  and  $s^2$ . Using (215) we have,

$$E \simeq E_0 + p_0 \left( \frac{n}{n_0} - 1 \right) + \frac{1}{2}(\lambda_0 + \frac{2}{3}\mu_0) \left( \frac{n}{n_0} - 1 \right)^2 + \mu_0 s^2. \quad (225)$$

Due to the relations (224), which imply that  $\text{Tr } \mathbf{s}$  is quadratic in  $\mathbf{e}$  to lowest order, it follows that up second order in  $\mathbf{e}$ , the scalar  $s^2$  is expressible, but actually not *uniquely* expressible, in terms of linear and quadratic invariants of  $\mathbf{s}$ . Any function of the form  $s^2 \simeq a \text{Tr } \mathbf{s} + (1-a) \text{Tr } \mathbf{s}^2 + b(\text{Tr } \mathbf{s})^2$  will give the same energy contribution up to that order. For the time being we simply assume that  $s^2$  is any such function and postpone a discussion of how to make an appropriate choice of it until later. To interpret eq. (225) physically, we note that the first three terms, i.e.

$$E_0 + p_0 \left( \frac{n}{n_0} - 1 \right) + \frac{1}{2}(\lambda_0 + \frac{2}{3}\mu_0) \left( \frac{n}{n_0} - 1 \right)^2 \quad (226)$$

gives the compression part, or “perfect fluid part”, of the equation of state, while the last term  $\mu_0 s^2$  adds to the equation of state a resistance against shear, *i.e.* against matter deformations that are not pure volume changes. The strength of this resistance is given by the shear modulus  $\mu_0$ . To extend this local expansion of the elastic matter equation of state into an equation of state valid for all admissible particle densities  $n$ , without neither introducing a third invariant nor inserting a nonlinear dependence on  $s^2$ , we simply assume that the energy per particle  $\epsilon$  is a first order polynomial in  $s^2$  with coefficients of arbitrary  $n$ -dependence. Following our previous convention that quantities depending only on  $n$  are checked, we have

$$\epsilon = \check{\epsilon}_0 + \check{\epsilon}_1 s^2. \quad (227)$$

Since the expansion of  $\epsilon$  to second order in  $\mathbf{e}$  is

$$\epsilon \simeq \check{\epsilon}_0(n_0) + \check{\epsilon}'_0(n_0)(n - n_0) + \frac{1}{2}\check{\epsilon}''_0(n_0)(n - n_0)^2 + \check{\epsilon}_1(n_0)s^2 \quad (228)$$

we can immediately see that the values of the expansion coefficients at  $n = n_0$  are

$$E_0 = n_0 \check{\epsilon}_0 \quad (229)$$

$$p_0 = n_0^2 \check{\epsilon}'_0(n_0) \quad (230)$$

$$\lambda_0 = n_0^3 \check{\epsilon}''_0(n_0) - \frac{2}{3}\mu_0 \quad (231)$$

$$\mu_0 = n_0 \check{\epsilon}_1(n_0) \quad (232)$$

Since these relations hold for any  $n_0$ , we may introduce  $n$ -dependent elasticity parameters according to

$$\begin{aligned} \check{\rho} &= n \check{\epsilon}_0 \\ \check{p} &= n^2 \check{\epsilon}'_0 \\ \check{\lambda} &= n^3 \check{\epsilon}''_0 - \frac{2}{3}\check{\mu} \\ \check{\mu} &= n \check{\epsilon}_1. \end{aligned} \quad (233)$$

The parameters  $\check{\rho}$ ,  $\check{p}$  and  $\check{\mu}$  will be called the *unsheared energy density*, the *unsheared pressure* and the *shear modulus*, respectively. The parameter  $\check{\lambda}$ , whose value at  $\check{p} = 0$  is one of the classical Lamé coefficients, does not have a useful physical interpretation and will not be mentioned by us elsewhere. Instead, the information given by the second derivative of  $\check{\epsilon}_0$  will be described by the *bulk modulus*

$$\check{\beta} = n \frac{d\check{p}}{dn} = n^3 \check{\epsilon}''_0 + 2\check{p} = \check{\lambda} + 2\check{p} + \frac{2}{3}\check{\mu}. \quad (234)$$

Let us now discuss the choice of shear invariant  $s^2$  which so far only is defined up to second order in  $\mathbf{e}$ . The primary requirements that we wish to impose are:

- $s^2$  should locally, around  $h_{ab} = {}^0\eta_{ab}$  for any admissible value of  $n_0$ , coincide with  $\text{Tr} \left\{ \left[ \mathbf{e} - \frac{1}{3}(\text{Tr } \mathbf{e}) \mathbb{1} \right]^2 \right\}$  up to second order in  $\mathbf{e}$ .
- $s^2$  should be positive definite in the sense that it should be strictly positive except at zero shear where it should vanish.
- $s^2$  should be an exact invariant of  $\boldsymbol{\eta}$ .

According to (224) a natural choice is to simply replace the locally defined shear matrix  $\mathbf{e}$  in (223) by either of the globally defined constant volume shear matrices  $\mathbf{s}$  or  $\hat{\mathbf{s}}$ , i.e. by either

$$\text{Tr} \left\{ \left[ \mathbf{s} - \frac{1}{3}(\text{Tr } \mathbf{s}) \mathbb{1} \right]^2 \right\} = \frac{1}{4} \text{Tr} \left\{ \left[ \boldsymbol{\eta} - \frac{1}{3}(\text{Tr } \boldsymbol{\eta}) \mathbb{1} \right]^2 \right\} = \frac{1}{4} n^{-4/3} \text{Tr} \left\{ \left[ \mathbf{k} - \frac{1}{3}(\text{Tr } \mathbf{k}) \mathbb{1} \right]^2 \right\} \quad (235)$$

or

$$\text{Tr} \left\{ \left[ \hat{\mathbf{s}} - \frac{1}{3}(\text{Tr } \hat{\mathbf{s}}) \mathbb{1} \right]^2 \right\} = \frac{1}{4} \text{Tr} \left\{ \left[ \boldsymbol{\eta}^{-1} - \frac{1}{3}(\text{Tr } \boldsymbol{\eta}^{-1}) \mathbb{1} \right]^2 \right\} = \frac{1}{4} n^{4/3} \text{Tr} \left\{ \left[ \mathbf{k}^{-1} - \frac{1}{3}(\text{Tr } \mathbf{k}^{-1}) \mathbb{1} \right]^2 \right\} \quad (236)$$

Indeed, the latter is the choice made by Carter and Quintana. However, we will choose a different definition. Both of the above expressions have rather cumbersome fractional power dependences on the linear particle densities  $n_\mu$ . We found this to lead to unnecessarily inconvenient formulae when using one of those in practice. The fractional powers are clearly due to the fact that the invariants are of second degree in  $\boldsymbol{\eta}$  or  $\boldsymbol{\eta}^{-1}$  whereas we are in a three-dimensional space. This suggests that a third degree invariant should be more suited from this point of view. The following choice turns out to be a suitable candidate,

$$s^2 = \frac{1}{36} [(\text{Tr } \boldsymbol{\eta})^3 - \text{Tr}(\boldsymbol{\eta}^3) - 24] = \frac{1}{36} n^{-2} [(\text{Tr } \mathbf{k})^3 - \text{Tr}(\mathbf{k}^3) - 24 \det \mathbf{k}] \quad (237)$$

Remarkably, this invariant looks precisely the same expressed in  $\boldsymbol{\eta}^{-1}$ , *i.e.* it is symmetric under  $\boldsymbol{\eta} \leftrightarrow \boldsymbol{\eta}^{-1}$ . It is not at all obvious that it fulfills the first two of the three requirements listed above. That the first requirement holds true can be proved in a straightforward manner. In order to show that our  $s^2$  fulfills the second requirement of positive definiteness we simply write down the invariant in terms of the linear particle densities. It is then given by

$$\begin{aligned} s^2 &= \frac{1}{36 (n_1 n_2 n_3)^2} [(n_1^2 + n_2^2 + n_3^2)^3 - (n_1^6 + n_2^6 + n_3^6) - 24 (n_1 n_2 n_3)^2] \\ &= \frac{1}{12} \left[ \left( \frac{n_1}{n_2} \right)^2 + \left( \frac{n_2}{n_1} \right)^2 + \left( \frac{n_2}{n_3} \right)^2 + \left( \frac{n_3}{n_2} \right)^2 + \left( \frac{n_3}{n_1} \right)^2 + \left( \frac{n_1}{n_3} \right)^2 - 6 \right] \\ &= \frac{1}{12} \left[ \left( \frac{n_1}{n_2} - \frac{n_2}{n_1} \right)^2 + \left( \frac{n_2}{n_3} - \frac{n_3}{n_2} \right)^2 + \left( \frac{n_3}{n_1} - \frac{n_1}{n_3} \right)^2 \right]. \end{aligned} \quad (238)$$

Clearly, the last expression shows that our  $s^2$  is positive except at  $n_1 = n_2 = n_3$ , *i.e.* at vanishing shear  $\boldsymbol{\eta} = \mathbb{1}$ . Let us finally consider the degenerate case of two equal particle densities. In section 7 we saw that setting  $n_2 = n_3$  gives

$$s^2 = \frac{1}{6} (z^{-1} - z)^2 \quad (239)$$

where  $z = n_1/n_2$ . Had we instead chosen the definition of  $s^2$  to be given by eq. (235), the corresponding expression would be

$$s^2 = \frac{1}{6} z^{2/3} (z^{-1} - z)^2, \quad (240)$$

or, using eq. (236),

$$s^2 = \frac{1}{6} z^{-2/3} (z^{-1} - z)^2. \quad (241)$$

For this degenerate case it is clear that the choice (237) has the simplest dependence on the particle densities.

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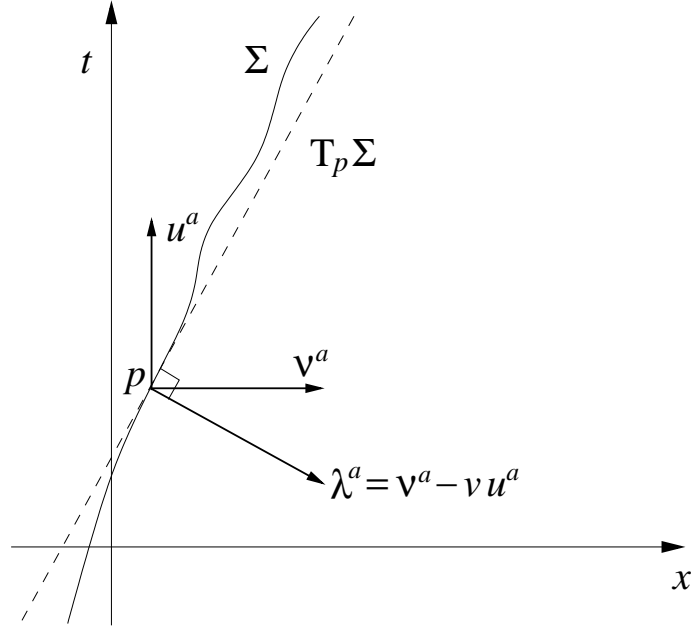


Figure 1: This schematic picture shows the wave front  $\Sigma$  moving in spatial direction  $\nu^a$  with speed  $v$  with respect to the flow  $u^a$  at a point  $p$ . It is convenient to introduce the vector  $\lambda^a = \nu^a - v u^a$  which is normal to  $\Sigma$  and gives the wave propagation speed as its contraction with  $u^a$ . Also sketched by a dashed line is the tangent space of  $\Sigma$  at  $p$ .

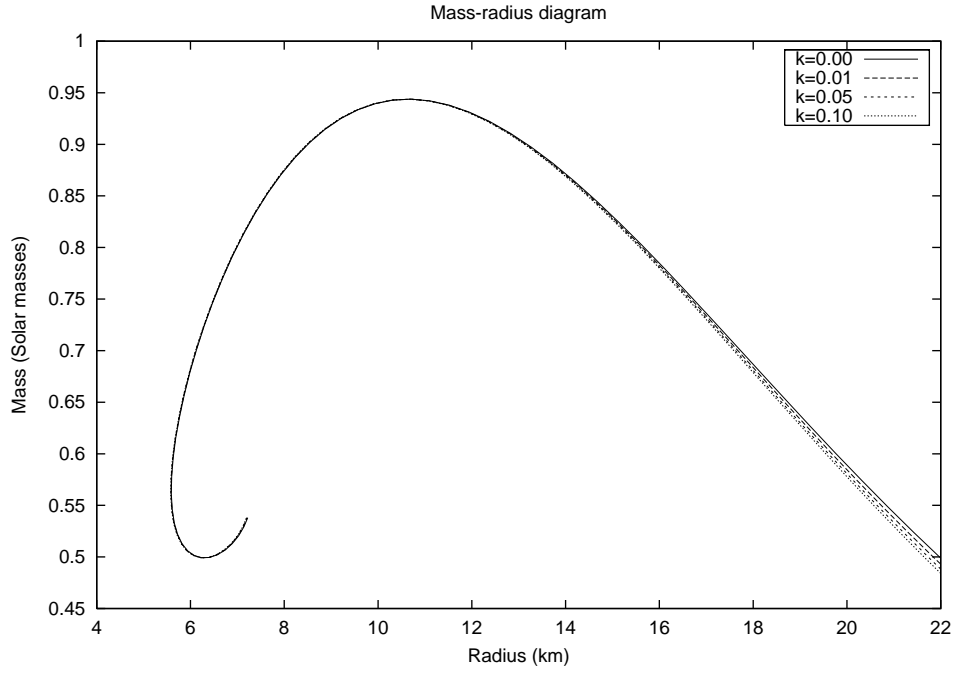


Figure 2: Mass-radius diagram for the models with a solid crust.

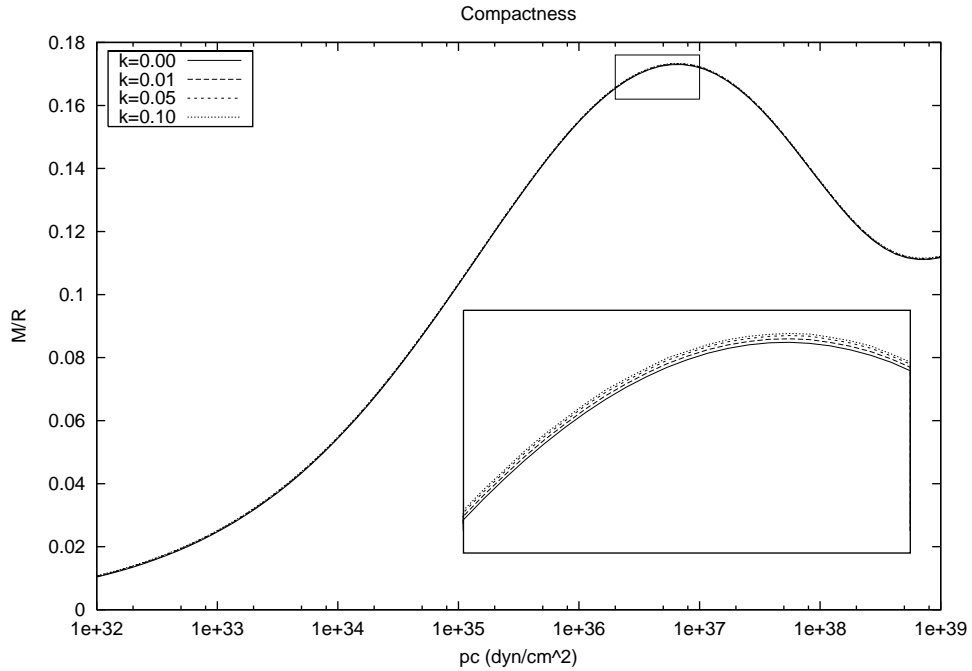


Figure 3: Compactness as a function of central pressure for models with a solid crust. Inserted is a magnification of a portion of the graph. There it can be seen that the compactness increases with increasing shear modulus, although the effect is very small.

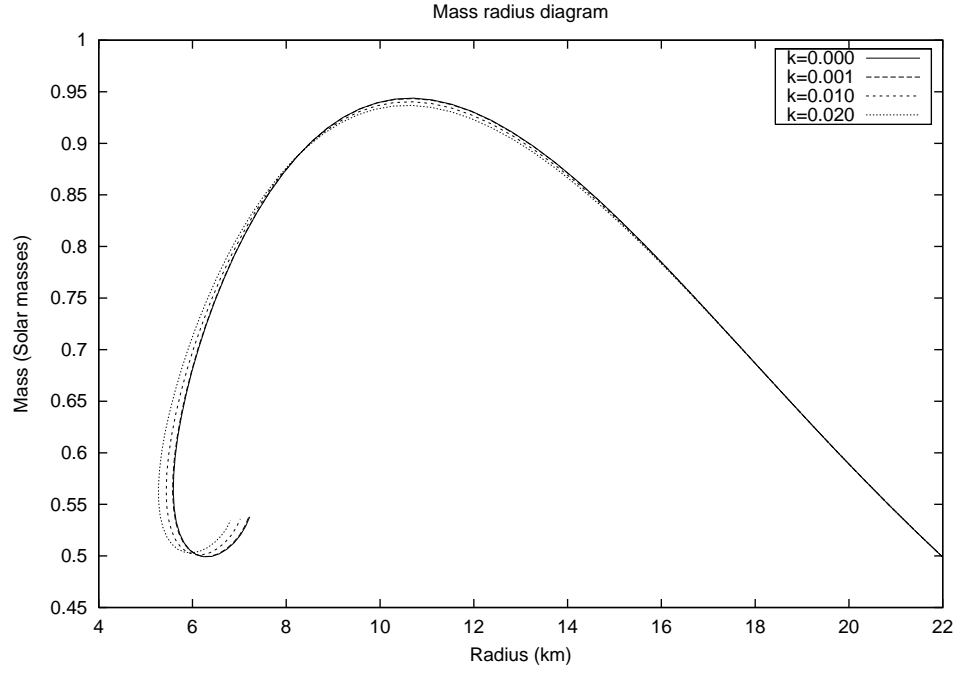


Figure 4: Mass radius diagram for models with a solid core.

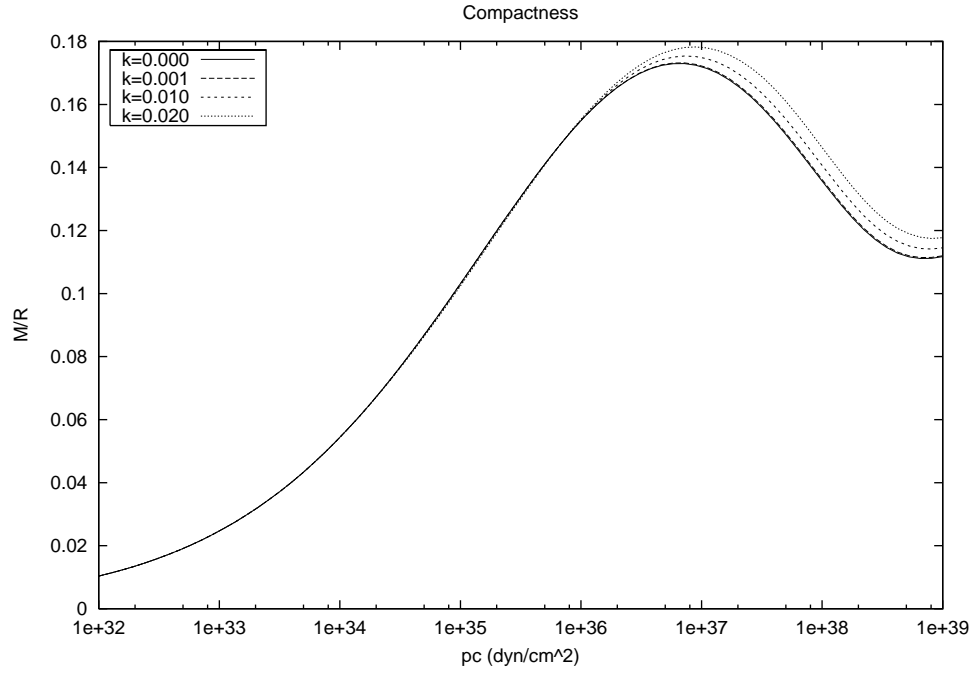


Figure 5: Compactness as a function of central pressure for models with a solid core.